Arthur T. Benjamin is Professor of Mathematics at Harvey Mudd College, where he has taught since 1989. He earned a B.Sc. from Carnegie Mellon University in 1983 and a Ph.D. in Mathematical Sciences from Johns Hopkins University in 1989. The Mathematical Association of America honored him with regional and national awards for distinguished teaching in 1999 and 2000 and named him the George Pólya Lecturer for 2006–2008.

Guidebook Contents

Part 2 of 2

Lecture 13: The Joy of Trigonometry
Lecture 14: The Joy of the Imaginary Number *i*
Lecture 15: The Joy of the Number *e*
Lecture 16: The Joy of Infinity
Lecture 17: The Joy of Infinite Series
Lecture 18: The Joy of Differential Calculus
Lecture 19: The Joy of Approximating with Calculus
Lecture 20: The Joy of Integral Calculus
Lecture 21: The Joy of Pascal's Triangle
Lecture 22: The Joy of Probability
Lecture 23: The Joy of Mathematical Games
Lecture 24: The Joy of Mathematical Magic

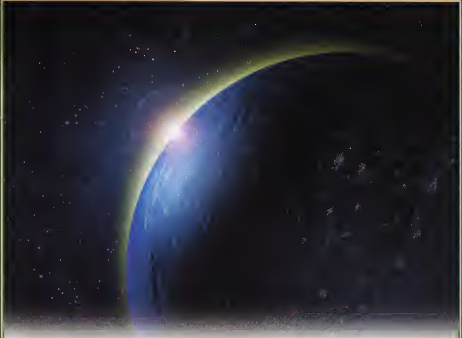
THE TEACHING COMPANY®
4151 Lafayette Center Drive, Suite 100
Chantilly, VA 20151-1232
Phone: 1-800-TEACH-12 (1-800-832-2412)
Fax: 703-378-3819
www.TEACH12.com

Cover Image: Earth Eclipsing the Sun © Digital Art/CORBIS

1411-02

[redacted]HE GREAT COURSES®

Science & Mathematics



The Joy of Mathematics

Taught by: Professor Arthur T. Benjamin
Harvey Mudd College

Part 2

Course Guidebook

[redacted] TEACHING COMPANY®

Arthur T. Benjamin, Ph.D.

Professor of Mathematics, Harvey Mudd College

Arthur T. Benjamin is a Professor of Mathematics at Harvey Mudd College. He graduated from Carnegie Mellon University in 1983, where he earned a B.S. in Applied Mathematics with university honors. He received his Ph.D. in Mathematical Sciences in 1989 from Johns Hopkins University, where he was supported by a National Science Foundation graduate fellowship and a Rufus P. Isaacs fellowship. Since 1989, Dr. Benjamin has been a faculty member of the Mathematics Department at Harvey Mudd College, where he has served as department chair. He has spent sabbatical visits at Caltech, Brandeis University, and University of New South Wales in Sydney, Australia.

In 1999, Professor Benjamin received the Southern California Section of the Mathematical Association of America (MAA) Award for Distinguished College or University Teaching of Mathematics, and in 2000, he received the MAA Deborah and Franklin Tepper Haimo National Award for Distinguished College or University Teaching of Mathematics. He was named the 2006–2008 George Pólya Lecturer by the MAA.

Dr. Benjamin's research interests include combinatorics, game theory, and number theory, with a special fondness for Fibonacci numbers. Many of these ideas appear in his book (co-authored with Jennifer Quinn), *Proofs That Really Count: The Art of Combinatorial Proof* published by the MAA. In 2006, that book received the Beckenbach Book Prize by the MAA. Professors Benjamin and Quinn are the co-editors of *Math Horizons* magazine, published by MAA and enjoyed by more than 20,000 readers, mostly undergraduate math students and their teachers.

Professor Benjamin is also a professional magician. He has given more than 1,000 "mathemagics" shows to audiences all over the world (from primary schools to scientific conferences), where he demonstrates and explains his calculating talents. His techniques are explained in his book *Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks*. Prolific math and science writer Martin Gardner calls it "the clearest, simplest, most entertaining, and best book yet on the art of calculating in your head." An avid games player, Dr. Benjamin was winner of the American Backgammon Tour in 1997.

Professor Benjamin has appeared on dozens of television and radio programs, including the *Today Show*, CNN, and National Public Radio. He has been featured in *Scientific American*, *Omni*, *Discover*, *People*, *Esquire*, *The New York Times*, *The Los Angeles Times*, and *Reader's Digest*. In 2005, *Reader's Digest* called him "America's Best Math Whiz."

Table of Contents
The Joy of Mathematics
Part II

Professor Biography		i
Course Scope		1
Lecture Thirteen	The Joy of Trigonometry	3
Lecture Fourteen	The Joy of the Imaginary Number *i*	8
Lecture Fifteen	The Joy of the Number *e*	14
Lecture Sixteen	The Joy of Infinity	19
Lecture Seventeen	The Joy of Infinite Series	24
Lecture Eighteen	The Joy of Differential Calculus	29
Lecture Nineteen	The Joy of Approximating with Calculus	[illegible]
Lecture Twenty	The Joy of Integral Calculus	41
Lecture Twenty-One	The Joy of Pascal's Triangle	47
Lecture Twenty-Two	The Joy of Probability	[illegible]
Lecture Twenty-Three	The Joy of Mathematical Games	[illegible]
Lecture Twenty-Four	The Joy of Mathematical Magic	[illegible]
Glossary		Part I
Bibliography		[illegible]

The Joy of Mathematics

Scope:

For most people, mathematics is little more than counting: basic arithmetic and bookkeeping. People might recognize that numbers are important, but most cannot fathom how anyone could find mathematics to be a subject that can be described by such adjectives as *joyful*, *beautiful*, *creative*, *inspiring*, or *fun*. This course aims to show how mathematics—from the simplest notions of numbers and counting to the more complex ideas of calculus, imaginary numbers, and infinity—is indeed a great source of joy.

Throughout most of our education, mathematics is used as an exercise in disciplined thinking. If you follow certain procedures carefully, you will arrive at the right answer. Although this approach has its value, I think that not enough attention is given to teaching math as an opportunity to explore creative thinking. Indeed, it's marvelous to see how often we can take a problem, even a simple arithmetic problem, solve it lots of different ways, and always arrive at the same answer. This internal consistency of mathematics is beautiful. When numbers are organized in other ways, such as in Pascal's triangle or the Fibonacci sequence, then even more beautiful patterns emerge, most of which can be appreciated from many different perspectives. Learning that there is more than one way to solve a problem or understand a pattern is a valuable life lesson in itself.

Another special quality of mathematics, one that separates it from other academic disciplines, is its ability to achieve absolute certainty. Once the definitions and rules of the game (the rules of logic) are established, you can reach indisputable conclusions. For example, mathematics can prove, beyond a shadow of a doubt, that there are infinitely many prime numbers and that the Pythagorean theorem (concerning the lengths of the sides of a right triangle) is absolutely true, now and forever. It can also "prove the impossible," from easy statements, such as "The sum of two even numbers is never an odd number," to harder ones, such as "The digits of pi (π) will never repeat." Scientific theories are constantly being refined and improved and, occasionally, tossed aside in light of better evidence. But a mathematical theorem is true forever. We still marvel over the brilliant logical arguments put forward by the ancient Greek mathematicians more than 2,000 years ago.

From backgammon and bridge to chess and poker, many popular games utilize math in some way. By understanding math, especially probability and combinatorics (the mathematics of counting), you can become a better game player and win more.

Of course, there is more to love about math besides using it to win games, or solve problems, or prove something to be true. Within the universe of numbers, there are intriguing patterns and mysteries waiting to be explored. This course will reveal some of these patterns to you.

In choosing material for this course, I wanted to make sure to cover the highlights of the traditional high school mathematics curriculum of algebra, geometry, trigonometry, and calculus, but in a nontraditional way. I will introduce you to some of the great numbers of mathematics, including π, e, i, 9, the numbers in Pascal's triangle, and (my personal favorites) the Fibonacci numbers. Toward the end of the course, as we explore notions of infinity, infinite series, and calculus, the material becomes a little more challenging, but the rewards and surprises are even greater.

Although we will get our hands dirty playing with numbers, manipulating algebraic expressions, and exploring many of the fundamental theorems in mathematics (including the fundamental theorems of arithmetic, algebra, and calculus), we will also have fun along the way, not only with the occasional song, dance, poem, and lots of bad jokes, but also with three lectures exploring applications to games and gambling. Aside from being a professor of mathematics, I have more than 30 years experience as a professional magician and I try to infuse a little bit of magic in everything I teach. In fact, the last lesson of the course (which you could watch first, if you want) is on the [illegible] mathematical magic.

Mathematics is food for the brain. It helps you think [illegible] creatively and helps you look at the world from multiple [illegible] it comes in handy when dealing with numbers [illegible] shopping around for the best bargain or [illegible] read in the newspaper. But I hope that you [illegible] a new way to experience beauty [illegible] logical argument. Many people [illegible] of art, and mathematic [illegible] If Elizabeth Barrett [illegible] "How do I count [illegible]

Lecture Thirteen
The Joy of Trigonometry

Scope: We begin this lecture by using a right triangle to define sine, cosine, and tangent of any acute angle (an angle between 0° and 90°), along with their reciprocals, the cosecant, secant, and cotangent. We then use the unit circle to expand this definition to allow us to define these trigonometric functions for any angle. From the Pythagorean theorem, we prove that the square of the cosine plus the square of the sine is equal to 1 for any angle. We derive the formula for the cosine of the sum or difference of any two angles, leading to a host of other trigonometric delights. We end by examining the graphs of trigonometric functions and mention some of their applications.

Outline

I. *Trigonometry* comes from the Greek *trigonometria*—literally, the measurement of triangles. It allows us to calculate measurements pertaining to triangles that we could not easily do using standard geometry techniques.

   A. All of trigonometry is based on two important functions known as the *sine function* and the *cosine function*. We will initially define these in terms of a right triangle.

      1. We begin with a right triangle with one angle labeled a. The side that is opposite a is called the *opposite side*. The other side adjacent to a that isn't the hypotenuse is called the *adjacent side*.

      2. We define the sine of a (abbreviated as "sin a") to be the length of the opposite side divided by the length of the hypotenuse:

         sine = opposite/hypotenuse.

      3. The cosine of a (abbreviated as "cos a") is defined as the length of the adjacent side divided by the length of the hypotenuse:

         cosine = adjacent/hypotenuse.

      4. The third most commonly used trigonometric function is the tangent function, which is the sine divided by the cosine. Because sine is opposite/hypotenuse and cosine is adjacent/hypotenuse, the tangent of a (abbreviated as "tan a") is their quotient:

         tangent = opposite/adjacent.

   B. We can now calculate some trigonometric values. For instance, let's look at a classic right triangle with side lengths 3, 4, and (hypotenuse length) 5.

      1. If the side opposite angle a has length 4, then $\sin a = 3/5$, $\cos a = 4/5$, and $\tan a = 3/4$.

      2. Note that the complementary angle to a has a measure of $90 - a$, an angle whose sine is 4/5, cosine is 3/5, and tangent is 4/3.

3. It's no coincidence that the sine of the second angle is the cosine of the first angle, and the cosine of the second angle is the sine of the first angle. Those values come straight from the definition:
   $\sin(90 - a) = \cos a$, $\cos(90 - a) = \sin a$.

C. You should also be aware of three other trigonometric functions:

function	*reciprocal of function*	*can be written as*
secant	cosine	sec = 1/cos
cosecant	sine	csc = 1/sin
cotangent	tangent	cot = 1/tan

II. The definitions that we've looked at so far allow us to define the sine, cosine, and tangent only for angles between 0° and 90° because that's all we can fit in a right triangle. A more general view of trigonometric functions allows us to define these for any angle.

A. We begin with the unit circle, which has a radius of 1. The unit circle has the equation $x^2 + y^2 = 1$.

B. We draw an angle of measure a on the unit circle. Let's label the point that corresponds to angle a as (x, y). If we drop a line [illegible] to the x-axis, we create a right triangle. We know that the [illegible] of this triangle is x, the height is y, and the [illegible]

C. What is cos a for that triangle? [illegible] hypotenuse side is length 1 [illegible] $a = y/1 = y$.

D. If cos a [illegible] called (x, y) [illegible] and [illegible]

E. [illegible]

F. [illegible] 180 [illegible] from the [illegible]

[illegible]

[illegible] cos a, and sin(a) [illegible]

H. [illegible] That literally takes us full [illegible] will be exactly the same as they [illegible] $(a + 360) = \cos a$, and $\sin(a + 360) = \sin a$

I. [illegible] the set of points (x, y) that satisfies [illegible] (cos a, sin a) is on the unit circle, that means that [illegible] This famous formula is usually written as: [illegible], or simply as $\cos^2 + \sin^2 = 1$.

III. The box below shows some other angles for our trigonometric vocabulary.

sin 0 = 0, cos 0 = 1. (That is, at 0°, $y = 0$, and $x = 1$.)
sin 30 = 1/2, cos 30 = $\sqrt{3}/2$.
sin 60 = $\sqrt{3}/2$, cos 60 = 1/2.
sin 45 = $\sqrt{2}/2$, cos 45 = $\sqrt{2}/2$.
sin 90 = 1, cos 90 = 0.

A. Notice that we don't have to memorize the tangents because they are simply the sine values divided by the cosine values.

B. Note also that the *arc tangent* of 1 (the angle whose tangent is 1) is 45°. The *arc sine* of 1/2 (the angle whose sine is 1/2) is 30°.

IV. Now we're ready to look at some problems.

A. We see a right triangle with an angle of 30°. What are the lengths of the other two sides of this triangle?
   1. Let b be the length of the hypotenuse. Since sin 30 = 1/2 and sin 30 (opposite/hypotenuse) = $10/b$, then $10/b$ = ½. Thus, $b = 20$. The length of the hypotenuse is 20.
   2. To find the length of the other side a, we'll use the Pythagorean theorem. We know that $b = 20$, and because we're dealing with a right triangle, we also know that $10^2 + a^2 = b^2$. We just saw that b^2 is 20^2, or 400, which tells us that a^2 is 300; therefore, $a = \sqrt{300}$, or $10\sqrt{3}$, or approximately 17.3.

B. We have a base of length 26, a side of length 21, and an angle of 15° between them. Can we find the area of this triangle?
   1. First, we'll draw a new line, splitting the triangle into two right triangles. The opposite here has height h and the hypotenuse has length 21; thus, sin 15 = h/21. Hence, h = 21 sin 15, and from our calculator sin 15 = .2588, so h is approximately 5.435.
   2. Knowing the height and the length of the base (given as 26), we can find the area of the triangle: ½bh, or ½ (26)(5.435) = 70.66.

V. We'll now prove one of the most difficult identities in basic trigonometry using a tool from our geometry lecture. (Keep in mind that you may have to go through this proof more than once before it sinks in.) This identity is as follows: $\cos(a - b) = \cos a \cos b + \sin a \sin b$.

A. Here's the tool from geometry: For a line of length L that goes from point (x_1, y_1) to point (x_2, y_2), by the Pythagorean theorem, we showed that L^2 is equal to $(x_1 - x_2)^2 + (y_1 - y_2)^2$.

B. We start our proof by looking at the unit circle. Focus on the triangle whose vertices are the origin, the point (0, 0); the point (cos a, sin a); and the point (cos b, sin b). We know that two of the side lengths of that triangle are 1 because they are radii of the unit circle. We want to

calculate the length of the line L that connects ($\cos a$, $\sin a$) to ($\cos b$, $\sin b$).

C. From the L^2 formula, we see $L^2 = (\cos a - \cos b)^2 + (\sin a - \sin b)^2$. We next expand that equation. The first term expands to $\cos^2 a + \cos^2 b - 2\cos a \cos b$. The second term expands to $\sin^2 a + \sin^2 b - 2\sin a \sin b$. Simplifying, $\cos^2 a + \sin^2 a = 1$, and $\cos^2 b + \sin^2 b = 1$. The expression now reads: $2 - 2\cos a \cos b - 2\sin a \sin b$.

D. We now rotate the triangle so that the lower side is lying on the x-axis. Note that the lengths of the sides are still 1, and the length L hasn't changed either. The angle that we're looking at is angle a minus angle b, or $a - b$. What is the length of L?

1. Look at the change of the x-coordinates and the change of the y-coordinates. Because the side of the triangle is lying on the x-axis and has a length of 1, that lower point is (1, 0); because the upper point of the triangle corresponds to angle $a - b$, it has coordinates $(\cos(a-b), \sin(a-b))$.
2. According to the L^2 formula, we add the change in x-coordinates squared and the change in y-coordinates squared: $(\cos(a - b) - 1)^2 + (\sin(a - b) - 0)^2$.
3. When we expand that, we get: $\cos^2(a - b) + 1 - 2\cos(a - b) + \sin^2(a - b)$. This equation is not as messy as it looks because $\cos^2 + \sin^2 = 1$. Thus, we have: $2 - 2\cos(a - b)$.

E. Now, we have to equate the two expressions that we found for L^2: $2 - 2\cos(a - b) = 2 - 2\cos a \cos b - 2\sin a \sin b$. We divide everything by 2 to get the desired formula: $\cos(a - b) = \cos a \cos b + \sin a \sin b$

VI. Once we have that equation, we can prove many useful identities. (Any truth in trigonometry is typically called a *trigonometric identity*.)

A. For instance, look what happens when we set $a = 90°$: $\cos(90 - b) = \cos 90 \cos b + \sin 90 \sin b$. But if you memorize $\cos 90 = 0$ and $\sin 90 = 1$, that equation simplifies to: $\cos(90 - b) = \sin b$. We can calculate $\sin(90 - a)$, which is $\cos(90 - (90 - a))$, $= \cos a$. This shows that those formulas are true for *any* angle—not just for angles between 0 and 90 degrees.

B. We have a formula for $\cos(a - b)$, but what about $\cos(a + b)$? We simply replace b with $-b$, so that the formula reads $\cos(a - (-b)) = \cos(a)\cos(-b) + \sin(a)\sin(-b)$. But $\cos(-b)$ is the same as $\cos b$, and $\sin(-b)$ is the negative of $\sin b$. When we plug those in, we get the equation: $\cos(a + b) = \cos a \cos b - \sin a \sin b$.

C. When a and b are the same angle, we have the *double-angle formula*: $\cos(2a) = \cos^2 a - \sin^2 a$.

D. We can do similar calculations with the sine function and show that $\sin(a + b) = \sin a \cos b + \cos a \sin b$. In particular, when a and b are equal, this formula says that $\sin 2a = 2\sin a \cos a$.

VII. Instead of using degrees that go from 0 to 360, mathematicians use a measurement called *radians*, in which $360° = 2\pi$ radians. Hence 1 radian is $360/2\pi$ degrees, approximately 57°.

VIII. Because the graphs of trigonometric functions come from the unit circle, they have a nice periodic property. The sine and cosine functions can be combined to model almost any function that goes up and down in a periodic way, such as seasons, sound waves, and heartbeats.

IX. We'll close with the *law of sines* and the *law of cosines*. For any triangle, with angles A, B, C, and corresponding side lengths a, b, c:

law of sines	$(\sin A)/a = (\sin B)/b = (\sin C)/c$
law of cosines	$c^2 = a^2 + b^2 - 2ab \cos C$

A. The law of cosines can be thought of as a generalization of the Pythagorean theorem.

B. With the law of cosines, we can find the length of a missing side, C, in a given triangle. In our previous example, the remaining side had length c, which satisfies $c^2 = 26^2 + 21^2 - 2(26)(21)\cos 15$. Since $\cos 15 = .9659$, we get $c^2 = 62.2$, or c is approximately 7.89.

Reading:

I. M. Gelfand and M. Saul, *Trigonometry*.

Eli Maor, *Trigonometric Delights*.

Questions to Consider:

1. Although it is useful to memorize the values of sine and cosine for 0°, 30°, 45°, 60°, and 90°, they can be easily derived from basic geometry. Try to do so. Once you know these values, then you can derive exact values for many other angles, as well. Use the double-angle formula to determine the exact value of the sine, cosine, and tangent of 15°.
2. Prove the law of sines, which states that for any triangle with angles A, B, C, and corresponding side lengths a, b, c: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. Hint: To prove the first equality, draw a perpendicular line from vertex A to the line BC. Now compute $\sin(A)$ and $\sin(B)$ and compare your answers.

Lecture Fourteen

The Joy of the Imaginary Number *i*

Scope: Does $\sqrt{-1}$ have any useful interpretation? Believe it or not, it actually does, although mathematicians were very slow to accept it, which is why they came up with such names as *imaginary* and *complex* to describe these quantities. Imaginary numbers have a simple arithmetic that make them easy to add, subtract, multiply, divide, and even find square roots, cube roots, and more. Moreover, once we learn to think outside the box of the real line and see complex numbers in a two-dimensional world, we can understand them even better. Complex numbers also have deep connections with trigonometry. Once we see that connection, neither subject is nearly as complex as it initially seems.

Outline

I. Let's begin by thinking a bit about negative numbers.

A. The ancient Greeks refused to accept the existence of [illegible] numbers, but when we think about numbers on a [illegible] readily understand the concept of negative numbers. [illegible] how to add, subtract, multiply, and [illegible]

B. In the real world, negative [illegible] imagine that they do. [illegible] which we suppose [illegible] learn about the imaginary [illegible]

1. [illegible]

2. [illegible] we get *0i*

C. [illegible] with imaginary numbers?

1. [illegible] we simply solve as we would using [illegible] that is, the *i*'s would cancel, and the answer would be 3. As long as we don't divide by 0, we don't get into trouble. The solution to $i \div i$ is 1.

2. How about $1/i$? What would be the reciprocal of this imaginary number? We multiply that number by 1, but we write 1 as the fraction i/i. When we do that, we get i/i^2; i^2 is still -1 and $i \div (-1)$ gives us $-i$.

3. We can also do the problem $1/2i$ by just multiplying fractions: $1/2(1/i)$, or $1/2(-i)$, which is $-i/2$.

D. Addition and subtraction with imaginary numbers are also easy. For example, $3i + 2i = 5i$; $3i - 2i = 1i$, or i; $2i - 3i = -1i$, or $-i$.

E. How about $2 + i$? The answer is just $2 + i$. A number that's of the form $a + bi$ is called *complex*. A number such as $4i$ is also a complex number, but it's called an *imaginary number*, because the "real part" is 0. Even a number such as 7 is a complex number, but it also happens to be a real number; thus, 7 can be thought of as $7 + 0i$. (Another rule of arithmetic for imaginary numbers is $0i = 0$.)

II. Let's look at some arithmetic with complex numbers. Sample problems are shown in the table below.

Addition	$(2 + 5i) + (5 + 3i) = 7 + 8i$
Subtraction	$(2 + 5i) - (5 + 3i) = -3 + 2i$
Multiplication	$(2 + 5i)(3 + 4i) = 6 + 15i + 8i - 20 = -14 + 23i$
Division	$\frac{(2+5i)}{(3+4i)} \times \frac{(3-4i)}{(3-4i)} = \frac{(26+7i)}{(3^2+4^2)} = \frac{(26+7i)}{25}$ $(2 + 5i)/(3 + 4i)$

A. If we add $(2 + 5i) + (5 + 3i)$, the answer is $7 + 8i$. How about $(2 + 5i) - (5 + 3i)$? We subtract the real part, $(2 - 5) = -3$, then we subtract the imaginary part, $(5i - 3i) = 2i$; thus, we get $-3 + 2i$.

B. When we multiply imaginary numbers, we use FOIL, just as we did with polynomials. For the problem $(2 + 5i)(3 + 4i)$, we get: $2(3) = 6$, $5i(3) = 15i$, $2(4i) = 8i$, and $5i(4i) = 20i^2$. The only new idea here is that we can use the fact that $i^2 = -1$. Thus, we have $6 + 15i + 8i - 20$, which simplifies to $-14 + 23i$.

C. Here's another problem: $(3 + 4i)(3 - 4i)$. (The second quantity is called the *conjugate* of the first quantity; that is, $a + bi$ has the conjugate $a - bi$. When we use FOIL, we get $9 + 12i - 12i - 16i^2$, which simplifies to $9 - 16(-1)$, $9 + 16 = 25$.

D. Conjugates help make the division of complex numbers easier. For instance, notice that if we multiply any number of the form $a + bi$ by its conjugate, $a - bi$, we get $a^2 - b^2i^2$, but the $-b^2i^2$ becomes $+b^2$; thus, $(a + bi)(a - bi) = a^2 + b^2$.

1. If we want to find the

$$\left(\frac{1}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right) = \frac{a-bi}{a^2+b^2} \quad \$1{,}000\left(1+\frac{.06}{n}\right)^n$$ reciprocal of $a + bi$,

$1/a + bi$, we simply multiply both the top and the bottom by the conjugate $a - bi$, as shown at right. Notice that the denominator of this fraction is a real number.

2. In this way, we never have to have a complex number in the denominator; we can always eliminate complex numbers by

multiplying the numerator and the denominator by the conjugate of the denominator.

3. For example, to divide $(2 + 5i)/(3 + 4i)$, we simply multiply the top and the bottom by $3 - 4i$. When we do that multiplication, we get $6 + 15i - 8i - 20i^2 = 26 + 7i$ in the numerator and $3^2 + 4^2 = 25$ in the denominator. That is, $(2 + 5i)/(3 + 4i) = (26 + 7i)/25$.

III. We saw that real numbers exist on the real line, but complex numbers exist on what's called the *complex plane*. Think of the x-axis and the y-axis, as we've been using in geometry.

A. For instance, the number $1 + i$ will have a "real part" of 1, an x-coordinate of 1, and a y-coordinate (called i) of 1. To find $1 + i$, then, we go to the right 1 and up 1. Think of the y-axis as having the number 0 where the two axes meet. As we go up, we see $1i, 2i, 3i,\ldots$, and as we go down the imaginary axis, or y-axis, we see $-i, -2i, -3i,\ldots$.

B. Let's do a few more examples: For $2 + 2i$, we go to the right 2 and up 2. For $-2 + i$, we go to the left 2 and up 1. For $-3 - 2i$, we go to the left 3 and down 2.

C. For $2 + i$, we go to the right 2 and up 1. What happens if we multiply $2 + i$ by 1/2? Then, we'd have the number $1 + 1/2i$, which would be halfway along the line from 0 to the point $2 + i$. Thus, when we multiply by 1/2, the length of the line changes by 1/2.

D. Similarly, if we multiply $2 + i$ by 2, we get $4 + 2i$. The line, or *vector*, that goes from the origin, the point $0 + 0i$, to the point $4 + 2i$ will be twice as long as it was before.

E. When we multiply a complex number by a real number, the line expands by a factor of that real number. If we multiply by a positive number, the line still points in the same direction. If we multiply by a negative number, the line points in the opposite direction.

IV. We can "see" how to add two complex numbers, such as $a + bi$ and $c + di$ by looking at their pictures on the complex plane.

A. We see a line that goes from 0 to $a + bi$ and a line that goes from 0 to $c + di$. Those two lines can form the sides of a parallelogram. The top of the parallelogram is the point at which the sum of $a + bi$ and $c + di$ meet. In other words, we start at $a + bi$, then add the vector that goes to $c + di$ to get the sum.

B. Look again at the line that goes from 0 to the point $a + bi$. We can define the length of that line to be the length of the complex number.

1. The base of the triangle we see has length a and the height has length b. By the Pythagorean theorem, the hypotenuse of this right triangle will have length $\sqrt{a^2 + b^2}$; we define that to be the length of the complex number $a + bi$.

2. The angle near the origin of this triangle would be the angle associated with the complex number. That angle is measured counterclockwise from the x-axis.

3. We can "see" how to multiply complex numbers in much the same way. In this case, we use two simple rules: Multiply the lengths from the origin and add the angles. We see, for example, if a complex number $a + bi$ has with an angle of about 30° and length 4 and if the point $c + di$ has an angle of 120° and length 2, to obtain their product, we simply multiply the lengths ($4 \times 2 = 8$) and add the angles ($30° + 120° = 150°$). Hence, the product will be:

$$8(\cos 150 + i \sin 150) = \left(\frac{-\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right) = -4\sqrt{3} + \frac{i}{2}.$$

C. To summarize, we can add two points, such as $(a + bi)$ and $(c + di)$, by drawing a parallelogram. We can multiply those points by multiplying the lengths and adding the angles.

D. Here's another example: $(2 + 2i)(-5 + 5i)$.

1. What's the length of the line from the origin to $2 + 2i$? The length of $a + bi$ is $\sqrt{a^2 + b^2}$; thus, the length of $2 + 2i$ will be $\sqrt{2^2 + 2^2}$, or $\sqrt{8}$. The length of $5 + 5i$ will be $\sqrt{5^2 + 5^2}$, or $\sqrt{50}$. When we multiply those lengths together, we get $\sqrt{400} = 20$.

2. The angle that cuts the first quadrant exactly in half at the point $2 + 2i$ is 45°. The angle for $-5 + 5i$ is 135°. When we add those angles together, we get 180°.

3. Incidentally, as we mentioned in the trigonometry lecture, a mathematician would call the measure of the first angle $\pi/4$ radians, instead of 45°. The second angle would be $3\pi/4$ radians, instead of 135°. Adding those together, we get $4\pi/4$ radians, or π radians.

4. Returning to the problem, when we multiply those numbers together, we get something that has a length of 20 and an angle of 180°. But 180° means 180° from the origin. Thus, we have a length of 20 pointing in the negative direction, or -20, as the answer.

E. Why does this rule of multiplying the lengths and adding the angles work? Once again, Euler gives us the equation for this: $e^{i\theta} = \cos\theta + i \sin\theta$ (e is a special number that we'll talk about later).

1. Look at the unit circle again. Euler says that we can simplify the point on the unit circle at angle θ can be called $e^{i\theta}$. Note that we would normally call that point $\cos\theta$, $\sin\theta$ if we were in the x, y plane, but in the complex plane, we call it $\cos\theta + i \sin\theta$, and we can simplify that to $e^{i\theta}$.

2. Any complex number on the unit circle is of the form $e^{i\theta}$. We can even get beyond the unit circle—that is, a point that has angle θ but has length R, represented by $Re^{i\theta}$.
3. For example, the number $2 + 2i$ has a length of $\sqrt{8}$ and an angle of $\pi/4$ radians. We can write this in *polar form* by saying $2 + 2i = \sqrt{8}e^{i\pi/4}$.
4. What if we were to stay on the unit circle and move 90°, or $\pi/2$ radians? We then find ourselves at the point i, that is, $e^{i\pi/2}$.
5. What happens when we multiply complex numbers? If we write those numbers in polar form—let's say our first number was $R_1e^{i\theta_1}$ and our second number was $R_2e^{i\theta_2}$ —then when we multiply those, we get $R_1R_2e^{i(\theta_1+\theta_2)}$. We're just using the laws of arithmetic and the law of exponents. The result, $R_1R_2e^{i(\theta_1+\theta_2)}$, tells us to do exactly what our two simple rules say, namely, multiply the lengths and add the angles.

F. What would Euler say about $(i)(i)$?

1. We said that $i = (e^{i\pi/2})(e^{i\pi/2})$; that would give us $e^{i\pi}$, but we also know that $(i)(i) = -1$; thus, $e^{i\pi} = -1$. That says that e^i multiplied by the angle of π radians (that's 180°) puts us at the real number -1.
2. If we rearrange that equation, it becomes: $e^{i\pi} + 1 = 0$, one simple equation that contains the five most important numbers and some of the most important relations in mathematics. What this "profound" equation says is simply that if you move 180° along the unit circle, you wind up at -1.

G. We can use Euler's equation to derive many complicated trigonometric identities. For example, we know $e^{i(2\theta)} = \cos(2\theta) + i\sin(2\theta)$. But it's also true that $e^{i(2\theta)} = e^{i\theta}e^{i\theta} = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$. Comparing the real and imaginary parts gives us $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, and $\sin(2\theta) = 2\sin\theta\cos\theta$.

V. Complex numbers can also help us with algebra.

A. For instance, without complex numbers, we could not find a solution to the equation $x^2 + 1 = 0$. We know, however, that this equation has at least one solution, namely, i, because $i^2 = -1 + 1 = 0$. We can also find another solution because $(-i)^2$ is also -1. Similarly, the equation $x^2 + 9 = 0$ has two solutions, namely, $3i$ and $-3i$, as does the equation $x^2 + 7 = 0$, which has solutions $\sqrt{7}i$ and $-\sqrt{7}i$.

B. With a more complicated algebraic expression, such as $x^2 + 2x + 5 = 0$, we use the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Plugging into that formula, we get the result shown below:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i .$$

C. Earlier, we discussed the fundamental theorem of algebra, but we can express that in a more polished way using complex numbers.

1. The fundamental theorem of algebra says that if $P(x)$ is a polynomial with real or complex coefficients, then we can always factor it in the form $P(x) = (x - r_1)(x - r_2)\ldots(x - r_n)$, where the roots $r_1, \ldots, r_n$ are complex numbers. We could factor any n^{th}-degree polynomial into these n parts.
2. We said earlier that the polynomial equation $P(x) = 0$ has, at most, n real solutions. In fact, we can say that it has "sort of exactly" n solutions, namely, $r_1, r_2, r_3, \ldots r_n$. (We say "sort of exactly" because it's possible that some of the roots were repeated.)

VI. In this lecture, we've defined imaginary numbers; seen how to add, subtract, multiply, and divide them; and how to use them algebraically and geometrically. We've also seen Euler's equation and some of its applications.

Reading:

Al Cuoco, *Mathematical Connections: A Companion for Teachers and Others*, chapter 3.

Paul J. Nahin, *An Imaginary Tale: The Story of* $\sqrt{-1}$.

Questions to Consider:

1. Once you overcome the obstacle of imagining $\sqrt{-1}$, it's easy to imagine the square root of any complex number. For instance, can you find two numbers with a square of i? (Hint: They will both lie on the unit circle.)
2. In general, for every positive integer n, every nonzero complex number has exactly n distinct n^{th} roots. For instance, can you describe all of the n^{th} roots of 1? Express your answer in terms $e^{i\theta}$ and plot the points on the unit circle. Prove that the sum of these roots is always 0.

Lecture Fifteen

The Joy of *e*

Scope: The irrational number $e = 2.71828182845904...$ is perhaps the most important number in calculus. Its importance stems from the fact that the *exponential function* e^x has the property that it is equal to its own derivative. In fact, any time the rate of growth of a function is proportional to the function's value, then e^x is inevitably present in that function. We'll also discuss the inverse of the exponential function, the natural logarithm. As we'll see, the number e also arises in a "tug-of-war" with 1 and infinity that allows us to compute compound interest and other important quantities with "ease."

Outline

I. Let's begin by creating e.

   A. We start with $(1 + 1/10)^{10}$. The result is 2.593…. Next, we look at $(1 + 1/100)^{100}$. We're doing two things here: We're making the base closer to 1, and we're making the exponent much bigger. The result for the second equation is 2.70481….

   B. Let's try again: $(1 + 1/1000)^{1000}$. That result is 2.71692…, still close to 2.7. In fact, as we take this process farther and farther out, as n gets larger and larger, $(1 + 1/n)^n$ gets closer and closer to the magical number e: 2.718281828459…. In mathematical terms, as n goes to infinity, e is the *limit* of $(1 + 1/n)^n$.

   C. We can generalize $(1 + 1/n)^n$: For any number x, if we take the limit as n goes to infinity of $(1 + x/n)^n$ we get e^x.

II. The number e relates to compound interest.

   A. Suppose you put \$1,000 in a bank account that earns 6% each year. After one year, how much money will you have? We can find the answer by multiplying \$1,000 by 1.06, which gives us \$1,060. Assuming that you didn't take the interest out of your account, after two years, you'll have $\$1{,}000 \times 1.06 \times 1.06$, or \$1,123.60. After three years, you'll have $\$1{,}000 \times 1.06^3$ = about \$1,191.02. After t years, you'll have $\$1{,}000 \times 1.06^t$.

   B. Let's focus on one year and suppose that instead of being compounded annually, the interest was compounded semiannually. Instead of giving you a lump sum of 6% at the end of the year, the bank gives you 3% after six months and another 3% when the year ends. That's equal to $\$1{,}000(1.03)^2$, or \$1,060.90.

   C. Suppose that your interest was compounded quarterly. That means that every three months, you'll get 1.5% interest. You figure the interest by $\$1{,}000(1.015)^4$. If the interest is compounded monthly, you get 0.5%

each month: $\$1{,}000(1.0005)^{12}$. If the interest is compounded daily, you figure the interest by: $\$1{,}000(1 + .06/365)^{365}$, which is \$1,061.83.

   D. If the bank compounds the interest *continuously*, your interest rate will be $6\%/n$ per time period. With \$1,000, you'll get:

      1. As we know from the formula we found for e^x, if we raise $(1 + .06/n)$ to the n^{th} power, as n gets larger and larger, we get closer and closer to $e^{.06}$. When we calculate $1{,}000 \times e^{.06}$, we get \$1061.84. Thus, with interest compounded continuously instead of daily, you earn an extra penny.

      2. But we also have a simpler equation than we had before. The general formula for 6% interest compounded continuously at the end of one year for \$1,000 is: $1{,}000(e^{.06})$. For t years, the formula is: $1000e^{.06t}$.

      3. Starting with a principal amount p and an interest rate r, after t years with continuous compounding, the general formula for interest is: pe^{rt}.

III. Let's do another application with e, this one involving homework.

   A. My students have turned in a number of homework assignments, but I don't want to grade them. I randomly return the homework to my students for grading, but I don't want any student to be in the position of grading his or her own paper. My question is: How likely is it that nobody gets his or her own paper?

   B. Suppose I have three students, A, B, and C. In how many ways can I return their homework papers? We know from our earlier lectures that there are $3! = 6$ ways of returning three homework papers, but only two out of the six ways result in no student getting his or her own homework back. Thus, if I randomly return the homework, the chance that no student gets his or her own homework is 2 out of 6.

   C. If I have four students, then there are $4! = 24$ ways of returning the homework papers. Of those 24 ways, only 9 result in no student getting his or her own homework back. The chances that no one gets his or her own homework are 9 out of 24, or 3/8, or .375.

   D. If we look at the chances with five students, six students, and so on, we see that the results get closer and closer to the same number. With five students, the chance is .366; with six students, it's about .368; with 100 students, it's .3678….

   E. Those results are strange. Whether I'm returning 100 papers back to 100 students, or 10 papers back to 10 students, or 1,000,000 papers back to 1,000,000 students, the chance that nobody gets his or her own homework is practically .368. This magic number .368 is $1/e$; it's the reciprocal of e, 2.71828.

   F. Why should this be? If I have n students in the classroom, the chance that the first student will get his or her own homework is $1/n$, and the chance is the same for the second student and so on. The chance that

you won't get your own homework back is $1 - 1/n$; therefore, the chances that no student gets his or her own homework are approximately $(1 - 1/n)^n$. Our earlier formula said that $(1 + x/n)^n$ approaches e^x as n gets large. That's the situation we have here except that x is -1. That is, we have $(1 + -1/n)^n$; as n goes to infinity, that result is e^{-1}, or $1/e$.

IV. The number e was first used by Isaac Newton, but it was studied, analyzed, and named by the great Swiss mathematician Leonhard Euler.

V. How does the function e^t grow?

A. Looking at the graph of that function, we see that e^t grows fairly quickly; e^{2t} grows faster, and e^{3t} grows even faster. These are called *exponential functions*.

B. Let's look at the function 5^t. The number 5 is between e, 2.718, and e^2, which is about 7.389. That means that 5^t is between e^t and e^{2t}; therefore, 5 is equal to e raised to some power between 1 and 2.

1. Let's say that 5 is e^r, in which r is some real number between 1 and 2. That means that we can replace 5 in the function 5^t with the number e^r raised to a power of t. Thus, 5^t is the same as $(e^r)^t$. By the law of exponents, that's e^{rt}.
2. To find the number r in this expression, we need to look at logarithms.

C. Logarithms are based on, initially, the powers of 10: $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1{,}000$, and so on. Negatively, $10^{-1} = 1/10$, $10^{-2} = 1/100$, and $10^{-3} = 1/1{,}000$. We say that the logarithm of x, denoted $\log x$, solves the equation $10^{\log x} = x$.

1. The logarithm of x is the exponent to which we have to raise 10 in order to get x. For example, $\log 1{,}000 = 3$ because $10^3 = 1{,}000$. Log $100 = 2$ because $10^2 = 100$. Log $10^y = y$ because we raise 10 to a power of y to get 10^y.
2. Can we find $\log\sqrt{10}$? The result for $\sqrt{10}$ is $10^{1/2}$; thus, $\log\sqrt{10}$ is 1/2.
3. What is log 512? A calculator tells us that log 512 = about 2.709. Does that seem reasonable? We know that log 100 = 2 and log 1,000 = 3. Because 512 is between 100 and 1,000, it follows that log 512 should also be between log 100 and log 1,000, or between 2 and 3.

D. There are other useful rules for logarithms. For instance, we've said that $\log 10^x = x$ for any x. Another sensible rule is $10^{\log x} = x$. Again, if we think about the definition of log, that makes sense.

E. Perhaps the most commonly used property of the logarithm is the one that states: The log of the product is the sum of the logs: $\log(xy) = \log x + \log y$.

1. Look at the expression $10^{\log x + \log y}$. According to the law of exponents, $10^{a+b} = 10^a \times 10^b$; thus, the expression would equal $10^{\log x} \times 10^{\log y}$. We know, however, that $10^{\log x}$ is x and $10^{\log y}$ is y, so that gives us $x \times y$. On the other hand, we know from our useful log rule that $10^{\log(xy)}$ is also equal to xy.
2. What have we done here? We've taken 10 to some power and obtained xy. We then took 10 to another power and obtained xy; therefore, the two powers must be equal. Equating these powers tells us that $\log x + \log y$ must equal $\log xy$.

F. As a corollary to that last rule, we can also show what I call the exponent rule: $\log(x^n) = n \log x$. Let's look at a couple of examples.

G. Historically, logarithms were useful for converting difficult multiplication problems into more straightforward addition problems.

H. Let's illustrate the product rule and exponent rule for logarithms.

1. If $\log 2 = .301\ldots$ and $\log 3 = .477\ldots$, then $\log 6 = \log(2 \times 3) = \log 2 + \log 3 = .301\ldots + .477\ldots = .778\ldots$.
2. Can we find log 5 knowing log 2 and log 3? We don't need to use log 3 in this solution, but we do need to use log 10, which is 1; thus, $\log 5 = \log(10 \times 1/2)$, or $\log 10 + \log 1/2$, and we know $\log 1/2$ because $1/2$ is 2^{-1}. We now have $\log 10 + \log 2^{-1}$, but by the exponent rule, $\log 2^{-1}$ is $-1 \times \log 2$. This is equal to $1 - \log 2$, or $1 - .301$, or about .699.
3. Earlier in this lecture, we looked at log 512. Note that 512 is 2^9. Log 2^9, by the law of log exponents, is equal to $9 \times \log 2$. Because log 2 is .301, that gives us 2.709, as we saw earlier.

I. We've been talking about logarithms using base 10, but we can also use logarithms in other bases. We define $\log_b x$ to be the exponent that solves $b^{\log_b x}$.

1. For instance, as we noted above 2^9 is 512; thus, the log(base 2) of 512 is 9 because we have to raise 2 to the 9th power to get 512.
2. The rules for logarithms in other bases are, in fact, virtually unchanged from the rules for base 10: $\log_b b^x = x$, $\log_b(xy) = \log_b x + \log_b y$, and $\log_b(x^n) = n\log_b x$.
3. We can also change from one base to any other base: Log(base b) x is $\log x \div \log b$, where that log could be the log(base 10) or any other base.
4. In chemistry and the physical sciences, the base 10 logarithm is probably the most popular. In computer science, base 2 is the most popular log. But in math, physics, and engineering, by far, the most popular base of the logarithm is the log base e, the natural log.

Reading:

Y. E. O. Adrian, *The Pleasures of Pi, e and Other Interesting Numbers*.

Eli Maor, *e: The Story of a Number*.

Questions to Consider:

1. With \$10,000 in a savings account earning 3% interest each year, compounded continuously, about how much money will be in the account after 10 years?
2. Starting with the famous formula for e: $1 + 1/1! + 1/2! + 1/3! + 1/4! + \ldots = e$, determine the following sums:

   $1/1! + 2/2! + 3/3! + 4/4! + 5/5! + \ldots$

   $1 + 3/2! + 5/4! + 7/6! + 9/8! + 11/10! + \ldots$

   $1/1! + 2/3! + 3/5! + 4/7! + 5/9! + 6/11! + \ldots$

Lecture Sixteen

The Joy of Infinity

Scope: What is the meaning of infinity? Are some infinite sets "more infinite" than others? What does it mean to sum an infinite number of numbers? These are the questions we'll explore in this lecture. At first glance, it would seem that there are more positive numbers than positive even numbers, yet both sets have infinite size. Moreover, these sets have the same *order of infinity* because they can be put in one-to-one correspondence. In the same way, we can show that there are as many fractions as positive integers, yet we will see that there are "more" irrational numbers than rational numbers. There is even an infinite number of levels of infinity. But we dare not explore these concepts too deeply, lest we meet the same fate as Georg Cantor, the man who developed these concepts but battled depression at the end of his life.

Outline

I. Is infinity a number?

   A. Technically, infinity is not a number. It's treated as if it's something larger than any number. As we go to the right on the number line, we're *approaching* infinity.

   B. Sometimes, though, we do treat infinity as a number, represented by the symbol ∞. For instance, we might say that adding all the positive numbers equals infinity, although most mathematicians would say that the sum *goes to* infinity.

   C. For the sum to go to infinity means that it will be larger than any number you ask for—larger than a million, a trillion, even a googol.

   D. Though it doesn't get as much attention, the cousin of infinity is negative infinity, denoted by $-\infty$. The sum of all negative numbers gets smaller than any negative number you could ask for.

   E. As a mathematical convenience, we make statements such as 1/infinity = 0. That makes sense because if we divide 1 by bigger and bigger numbers, then the quotient gets closer and closer to 0. We can even say $1/-\infty = 0$ because if we divide 1 by negative million, or negative billion, etc., the result gets closer to 0.

   F. On the other hand, we are never allowed to divide by 0; thus, we couldn't say $1/0 = \infty$. The real reason we don't allow that is because 1/0 could be infinity or negative infinity. If we divide 1 by a tiny positive number, the answer will be a big positive number. If we divide 1 by a tiny negative number, the answer will be a big negative number. In other words, as our denominator gets closer to 0 from the right, we're going to infinity; as it gets closer to 0 from the left, we're going to negative infinity. That's why we let 1/0 be undefined.

G. There are some infinite sums that add to something besides infinity. For instance, $1 + 1/2 + 1/4 + 1/8 + 1/16 + \ldots = 2$. Also, $1 + 1/1! + 1/2! + 1/3! + \ldots = e$. We don't necessarily get infinity as our answer even if we have an infinite sum.

II. In this lecture, rather than using infinity as a number-like object, we will use it as a size.

A. The size of a set (or *cardinality* of a set) S (denoted $|S|$) is the number of elements in the set. For instance, if $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then $|S| = 10$. What's the size of the set of all positive integers? Because that set has infinitely many elements, its size is infinity; the size of the set of all even numbers is also infinity.

B. What is the size of the set of all fractions? Because there are more fractions than there are integers, the size of that set is infinite as well. The size of the set of real numbers between 0 and 1 is also infinite.

C. As we will see, however, some infinities are more infinite than others.

III. Let's try a thought experiment.

A. Suppose that every chair in my class is occupied by a student, and no students are chair-less. I could pair up students with chairs and conclude that there are as many students as there are chairs. This is called a *one-to-one correspondence.*

B. We can use this same idea to compare the set of positive odd numbers with the set of positive even numbers. Not only are there an infinite number of both of those objects, but they have the same *order of infinity* because we can pair them up. Those sets, then, are infinite, and they have the same size.

C. What about the sizes of the sets of all positive integers (1, 2, 3, 4, 5, 6,…) and all positive even integers (2, 4, 6, 8, 10, 12,…)?

1. I claim that that those two sets have the same size—not because they are infinite, but because we can pair them up.
2. Here, 1 is paired with 2, 2 is paired with 4, 3 is paired with 6, and so on.

D. Mathematicians say that any set that can be paired up with the positive integers is *countable* because we could essentially list all the numbers in the set just by counting.

1. For example, the set of all integers (positive, negative, and 0) is countable. It can be put in one-to-one correspondence with the positive integers because we can list them all with no infinite gaps.
2. If we list the integers as 0, 1, –1, 2, –2, 3, –3, 4, –4, 5, –5…, eventually, we will reach every positive and every negative number. We can't, however, list the positive numbers first, then the negative numbers, because we'd never finish with the first step.

E. These ideas were first put forth by the German mathematician Georg Cantor, but it took mathematicians decades to come to grips with them.

IV. Now, let's look at a larger set of numbers, the set of fractions.

A. Of course, there are an infinite number of positive fractions, but are they countable? We might try listing the fractions out row by row, but that would leave infinite gaps. If we list the fractions out diagonally, however, we see that the set of rational numbers is countable. It has the same size as the set of positive integers.

B. Can we find a set that is not countable? Surprisingly, the set of real numbers between 0 and 1 is uncountable. We can show this with a proof by contradiction.

1. Suppose you begin your list with the number .31415926…; your second number is .12121212…; your third number is .500000; and your fourth number is .61803399…. I can use your list to create a real number that can't be on the list.
2. I begin with your first number, .31415926…. I add 1 to the first digit of that number to change it to 4. Then, I add 1 to the second digit of your second number to change that digit to 3. I can also change the third digit of your third number, the fourth digit of your fourth number, and so on. In that way, I create a number that is not on your list.
3. Let's say I created the number .4311…. How do I know that number is not the millionth number on your list? It couldn't be, because it will have a different digit in the millionth digit past the decimal point. The number I've created, then, can't be the first, the second, or the millionth number on your list.
4. Therefore any attempt to list the real numbers is doomed to failure: the list is guaranteed to be incomplete.

V. We know that the set of positive real numbers and the set of real numbers between 0 and 1 are both infinite, but the first set is countable and the second set is not. We now need to come up with different notations to represent these two different levels of infinity.

A. We use the symbol $\aleph_0$ ("alef nought") to denote the size of the set of positive integrers. (The symbol alef is the first letter of the Hebrew alphabet.) Anything that can be put into correspondence with the positive integers, any countable set, has size, or cardinality $\aleph_0$. The set of real numbers between 0 and 1 has a greater level of infinity; mathematicians usually denote that level of infinity by the letter c, where c stands for continuum.

B. Can we find a set that is bigger than c? For example, is there twice as much "stuff" in the interval between 0 and 2 as there is in the interval between 0 and 1? Both of these are infinite sets, but there is an elementary way to pair up the numbers between these two sets.

1. Let's look at a triangle. Inside the triangle, we have a segment of length 1 and, at the base, we have a segment of length 2. At the top, we have a laser beam shooting down, connecting every point between 0 and 1 with another point between 0 and 2. We can pair every point in the first interval with a point in the second interval.
2. What we're really looking at here is the function $y = 2x$. Every point on the x-axis is associated with a point on the y-axis by way of that function. This shows that the size of the set of real numbers between 0 and 2 is the same as the size of the set of real numbers between 0 and 1. In other words, both have size c.

C. What about the size of the set of all real numbers from negative infinity to positive infinity? Is that set bigger than the set between 0 and 1?

1. As long as we draw any function that more or less increases from negative infinity to positive infinity, we can create a one-to-one correspondence. The function we see here is a trigonometric function: $y = \tan(\pi(x - 1/2))$.
2. Between every number from 0 to 1, we can get every real number—positive, negative, and 0. In other words, the size of the set of real numbers is the same as the size of the set of real numbers between 0 and 1. Both still have size c.

D. Can we find a set that has a size bigger than c?

1. Let's look at the plane—that's the set of points inside the unit square (side length = 1). If there are an infinite number of points between 0 and 1, there are certainly an infinite number of points in the square that is drawn from 0 to 1 horizontally and from 0 to 1 vertically. Amazingly, however, even this set can be put in one-to-one correspondence with the set of real numbers between 0 and 1.
2. Let's say that x is $0.r_1r_3r_5r_7\ldots$, and y is $0.r_2r_4r_6r_8\ldots$. That's an ordered pair inside the unit square. We will associate that pair with the real number $0.r_1r_2r_3r_4r_5r_6\ldots$.
3. If we start with, say, the point 0.31415926…, that pairs up with the ordered pair 0.3452… and .1196…. Any number between 0 and 1 can be turned into a pair of numbers between 0 and 1, and vice versa.
4. To put it another way, the size of the set called $\mathbb{R}^2$ (the set of all pairs of real numbers; pronounced "R two") is c, where c stands for "continuum." The sizes of the sets of all triples, quadruples, and so on of real numbers are also c. We still haven't found a set that is bigger than the size of the set of real numbers.

E. There is such a larger set: the set of all curves in the plane. That is, there are more curves than there are real numbers to assign them.

F. Here is another set whose size is bigger than c: the set of all subsets of real numbers. That is, there are more subsets of real numbers than there are real numbers to assign them.

VI. In this lecture, we've shown that a set is infinite if the size of that set exceeds any given number. The sets of integers and rational numbers are countable because we can list them; these have a size called $\aleph_0$. The real numbers are uncountable and have size c. Finally, there are infinitely many levels of infinity. Here's a question to think about: Are those infinitely many levels of infinity countably infinite or uncountably infinite?

Reading:

Edward B. Burger and Michael Starbird, *The Heart of Mathematics: An Invitation to Effective Thinking*, chapter 3.

William Dunham, *Journey through Genius: The Great Theorems of Mathematics*, chapters 11–12.

Eli Maor, *To Infinity and Beyond: A Cultural History of the Infinite*.

Questions to Consider:

1. Prove that the number of irrational numbers between 0 and 1 is uncountable.
2. Imagine a red robot that produces 10 billiard balls at a time, numbered 1 though 10, then 11 through 20, then 21 through 30, and so on. Meanwhile, each time the red robot creates 10 balls, an evil green robot destroys a ball. In the first round, it destroys ball 10; in the second round, it destroys ball 20; in the third round, it destroys ball 30; and so on. At the end of the process, which balls remain? (Although this is an infinite process, we can imagine it happening in a finite amount of time. Imagine that round 1 occurs an hour before midnight, round 2 occurs half an hour before midnight, round 3 occurs a third of an hour before midnight, and so on. The challenge is to describe the situation "at midnight.")
3. Bonus question: now suppose instead that after the first round, the green robot destroys ball 1; after the second round, it destroys ball 2; after the third round, it destroys ball 3; and so on. At the end of this process, which balls remain?

Lecture Seventeen

The Joy of Infinite Series

Scope: What does it mean to add up an infinite number of numbers? In this lecture, we demonstrate that "some sums sum" to infinity. For example, the *harmonic series* 1 + 1/2 + 1/3 + 1/4 + 1/5 + ... gets arbitrarily large. But other sums stay small: for example, the *geometric series* 1 + 1/2 + 1/4 + 1/8 + 1/16 +... actually *equals* 2. More surprising still, some infinite series can be rearranged to obtain an entirely different sum.

Outline

I. Let's look at several proofs of the bold statement .999999... = 1.

A. Here's the most elementary proof: We agree that 1/3 = .333333333.... If we multiply 3 × .333333..., we get .999999.... We also know that 3 × .333333 = 3(1/3), but 3(1/3) is exactly equal to 1. If we follow that chain of logic, we get: .999999... = 3 × .333333... = 3(1/3) = 1.

B. Here's another proof: Let S = .999999... S; then $10S$ = 9.999999.... Subtracting, we get, $9S = 9$, hence $S = 1$.

C. Here's yet another proof: We agree that .999999... must be less than or equal to 1. That means that 1 – .999999... is greater than or equal to 0. But 1 – .999999.... would be 0.000000.... We can say that either that difference is 0 or that it's smaller than any positive number and, thus, must be 0. We have, then, two quantities, 1 and .999999..., whose difference is 0, and if two quantities have a difference of 0, they must be the same.

D. In summary, we could say that .99 is close to 1 and .999 is even closer to 1, but .999999... is as close to 1 as desired. And for that reason, we say that those quantities are equal.

E. Another way of looking at .999999... is as an infinite sum, the topic for this lecture. Technically, .999999... = .9 + .09 + .009 + .0009..., and we're interested in what happens when we add an infinite number of numbers together. In general, we say that a series, such as $a_1 + a_2 + a_3 + a_4 + \ldots$, has a sum of S if the sum gets arbitrarily close to S. As an example, .9 + .09 + .009 + ... gets arbitrarily close to 1.

II. Let's look at the example: 1 + 1/2 + 1/4 + 1/8 + 1/16 + ... = 2.

A. Imagine that the distance between me and a table is 2 feet. If I walk halfway toward the table, I've just walked 1 foot. If I walk half the distance again, I've walked 1/2 foot. If I walk half the distance again, I've walked 1/4 foot. With every step I take, I'm walking half as much as I did with the previous step. Technically, I never reach the table, but I get arbitrarily close to the table. That's why we say that the sum 1 + 1/2 + 1/4 + 1/8 + ... = 2. That sum gets as close to 2 as we desire.

B. As an infinite sum gets closer and closer to a single number, it is said to *converge*. If it doesn't converge, it is said to *diverge*.

1. For example, the sum we just looked at converges to 2. The earlier example, .9 + .09 + .009 + ..., converges to 1. In contrast, the sum 1 + 2 + 4 + 8 + 16 + ... diverges to infinity.
2. A sum can diverge without getting larger. For instance, the sum 1 – 1 + 1 – 1 + 1 – 1 ... is first 1, then 0, then 1 again, then 0, and so on. Because that sum is not getting closer to any real number, we say that it diverges.
3. In order for a sum to converge, the terms of the sum must get closer to 0; otherwise, the sum will not get closer to a real number.
4. For example, the series 1 + 1/2 + 1/4 + 1/8 + 1/16 + ... is known as a *geometric series*, which has the form $1 + x + x^2 + x^3 + x^4 \ldots$. In order for the terms to be getting closer to 0, the number x must be between –1 and +1.

C. Here is the formula for the geometric series: For any number x strictly between –1 and +1, the series $1 + x + x^2 + x^3 + x^4 + \ldots = 1/(1 - x)$. Let's look at a proof of that formula.

1. Let $S = 1 + x + x^2 + x^3 + x^4 \ldots$. Multiplying that equation by x, on the left, we have $x(S)$; on the right, we have $x + x^2 + x^3 + x^4 + \ldots$. Taking away the "excess," we have $S - xS$ on the left, or $S(1 - x)$; on the right, we're left with 1. Solving for S, we get $S = 1/(1-x)$.
2. Let's do the example we saw earlier: When $x = 1/2$, then1+ 1/2 + 1/4 + 1/8 + 1/16 + ... = $\dfrac{1}{1-1/2}$, but the denominator, 1 – 1/2, is equal to 1/2; the answer, then is $\dfrac{1}{1/2}$, which is 2.
3. When $x = -1/2$, the geometric series tells us that 1– 1/2 + 1/4 – 1/8 + 1/16 – ... = $\dfrac{1}{1-(-1/2)}$, or $\dfrac{1}{3/2}$, which is 2/3.

D. Let's go back to the number that we started with: .999999....

1. We can write that number as .9 + .09 + .009 + ... That's not a geometric series yet, but we can factor out a .9 from everything, leaving us with .9(1 + .01 + .001 + .0001 + ...).
2. Those terms are the quantity 1/10th raised to higher and higher powers. In other words, we've pulled out a factor of 9/10 and we're multiplying it by $1 + 1/10 + 1/10^2 + 1/10^3 + 1/10^4 + \ldots$.
3. Adding, that infinite series is $\dfrac{1}{1-1/10}$. In other words, we have 9/10(10/9), which is 1. That's our last proof of the fact that .999999... = 1.

E. When you use the formula for the geometric series, you must be careful that the x you're using is strictly between –1 and 1; if x is greater than or equal to 1 or less than or equal to –1, then the formula doesn't work.

For instance, if we let $x = 2$, then the geometric series produces the nonsensical result that $1 + 2 + 4 + 8 + 16 + \ldots = \frac{1}{1-2}$, which is -1.

III. Let's do an application of the geometric series.

A. Suppose a ball is dropped from a 50-foot building, and the ball always rebounds to 80% of the height from which it was dropped. How far does the ball travel?

B. Obviously, the ball goes 50 feet down originally, but then it travels up 80% of that, or 40 feet. Then, it drops 40 feet and rebounds up 80% of that, or 32 feet. It drops 32 feet, then rebounds up 25.6 feet, then drops 25.6 feet, and so on. What's the total amount that the ball travels?

C. We can write this out as a geometric series, as shown below.

$$50 + 2(50)(.8) + 2(50)(.8)^2 + 2(50)(.8)^3 + \ldots$$

Simplifying: $50 + 80(1 + .8 + .8^2 + .8^3 + \ldots)$

Solving: $50 + 80\left(\frac{1}{1-.8}\right) = 50 + 80\left(\frac{1}{1/5}\right) = 450$ ft

IV. If a sum $a_1 + a_2 + a_3 + \ldots$ converges, we know that its terms must go to 0, but does that guarantee that the sum converges? Surprisingly, the answer is no. We can understand this by looking at the *harmonic series*: $1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots$.

A. Before we look at this proof, note that the harmonic series was given its name by the ancient Greeks. They noticed that strings with lengths of 1, 1/2, 1/3, 1/4, and 1/5, and so on, when plucked, tended to produce harmony.

B. Now let's look at the proof that the harmonic series goes to infinity.

1. If we take $1/2 + 1/3 + \ldots + 1/9$, we're adding nine terms, and you would agree that each of those terms is bigger than 1/10. Thus, the sum of those nine terms must be at least 9/10.
2. Now, let's look at the next 90 terms, the numbers $1/10, 1/11, \ldots, 1/99$. We've just added 90 more terms, and each of those terms is bigger than 1/100; the sum of those 90 terms is at least 90(1/100), which is 9/10. Thus, the sum of those 90 terms is bigger than 9/10.
3. In the same way, each of the next 900 terms is bigger than 1/1,000, which means that each of those terms is bigger than 9/10. Then, the next 9,000 terms also add to something bigger than 9/10, and the next 90,000 terms add to something bigger than 9/10.
4. In this way, the sum of all these terms is bigger than $9/10 + 9/10 + 9/10 + \ldots$. This sum gets arbitrarily large; thus, it diverges to infinity.

C. Could we scale down the harmonic series somewhat? What if we cut down every term by 100? Does the sum $1/100 + 1/200 + 1/300 + 1/400 + \ldots$ converge or diverge?

1. We could factor 1/100 out of those terms, and we would be left with $1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots$, but we know that series diverges to infinity and 1/100 of infinity is still infinity.
2. Interestingly, increasing the denominators of the harmonic series slightly brings about enough of a change to get the series to converge. Instead of using the denominators 2, 3, 4..., we use $2^{1.01}$, $3^{1.01}$, and $4^{1.01}$. That makes the denominators a little bit bigger, which makes the fractions a little bit smaller, and the sum, then, will be less than infinity.

V. Let's now turn to what mathematicians call an *alternating series*.

A. We start with the numbers $a_1 > a_2 > a_3 > a_4 > \ldots > 0$. If these numbers are getting closer and closer to zero, then the sum of $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots$ will converge to a single number. For example, $1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \ldots$ must converge.

B. To prove this, think of starting at 1, then subtracting 1/2, then adding 1/3, then subtracting 1/4, adding 1/5, subtracting 1/6, adding 1/7, subtracting 1/8, and so on—getting closer and closer to a single point.

C. The sum is honing in on a single number, which incidentally, is .693..., the natural log of 2. The explanation for that, however, requires calculus.

D. Let's look again at the same series: $1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \ldots$. Notice that the denominators consist of all the positive numbers, and all the odd denominators are counted positively and all the even denominators are counted negatively. Knowing this, we can add that series up in a slightly different way.

1. Consider the series shown below:

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)-\frac{1}{16}+\left(\frac{1}{9}-\frac{1}{18}\right)-\frac{1}{20}+\ldots$$

Even though it looks different, this is just a rearrangement of the original series: every odd denominator is added once and every even denominator is subtracted once.

2. Next, we'll group those numbers, which results in:

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}+\frac{1}{18}-\frac{1}{20}+\ldots$$

That is equal to: $\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\ldots\right)$,

or half the original series.

3. We started with the series $1 - 1/2 + 1/3 - 1/4\ldots$, and when we rearranged it, we paradoxically wound up with half of the original

series. In fact, we can rearrange these same sets of numbers to obtain any sum we want.

4. The lesson here is that the commutative law, $a + b = b + a$, can fail when adding infinite numbers of positive and negative terms.

Reading:

Y. E. O. Adrian, *The Pleasures of Pi, e and Other Interesting Numbers.*

Daniel D. Bonar and Michael J. Khoury, *Real Infinite Series.*

Questions to Consider:

1. Prove $1/2 + 1/6 + 1/12 + 1/20 + 1/30 + ... = 1$, where the first denominator is 1×2, the second denominator is 2×3, and so on. (Hint: $1/12 = 1/3 - 1/4$).
2. Suppose that in the harmonic series, we throw away all terms with the number 9 in the denominator (i.e., we eliminate such numbers as 1/9, 1/19, 1/29, 1/97, 1/3141592, and so on). Show that this 9-less series converges.

Lecture Eighteen
The Joy of Differential Calculus

Scope: Calculus is the mathematics of how things change. In algebra, we learned to measure the slope of a straight line. But how do we measure slope on a curve or, more precisely, the slope of the tangent line at any point on a curve? By computing slopes of secant lines and working with the concept of a limit, we can answer this important question. Beginning with the derivative (rate of change) of the function $y = x^2$, we learn such rules as the power rule, product rule, and quotient rule that allow us to compute derivatives of polynomials and more complicated functions, including trigonometric functions.

Outline

I. The words *calculus* and *calculate* have the same root, which is *calculus*, literally meaning "pebble." Pebbles were the first calculating devices. In calculus, we learn to calculate how things grow and change. There are many applications for calculus, particularly in astronomy, physics, chemistry, economics, and engineering.
   A. In this first lecture on calculus, we'll have fun with functions, seeing how they grow and change over time.
   B. In the next lecture, we'll find an approach for approximating any function with a polynomial, the simplest of functions.
   C. In our third lecture on calculus, we'll explore the fundamental theorem of calculus, which allows us to calculate areas and volumes that are impossible to find using only the tools of geometry and trigonometry.

II. We begin with the study of slopes, which we encounter every day. Any time one quantity varies with another quantity, such as in calculating miles per gallon or price per pound, the idea of slope is involved.
   A. Mathematically, the simplest slopes are straight lines, where the slope is constant. For instance, we know from our earlier discussion of algebra that the function $y = 2x + 3$ produces a line with a slope of 2.
   B. The line for the function $y = 4x - 1$ has a much steeper slope, 4. The line $y = -x$ has a constant slope of -1. Finally, a constant function, which is also a straight line, such as $y = 5$, has a slope of 0.
   C. Lines have the same slope everywhere, but calculus applies our knowledge of lines to curves, which are not nearly as simple. For instance, let's look at the parabola $y = x^2 + 1$. It doesn't make any sense to try to find the slope of a parabola, because it's constantly changing. But we can ask how fast the function is growing at a specific point.
      1. When $x = 3$ on this graph, $y = 3^2 + 1$, which is 10. How fast is the function growing at the point (3, 10)? We're interested in the slope

of the line that just touches the graph at the point (3, 10). That line is called the *tangent line*, and our mission is to calculate the slope of that tangent line.

2. We need two points to figure out the slope, but we can use a point that's close to (3, 10) that also lies on the parabola. Let's look at $x = 3 + h$; the y value for that would be $(3 + h)^2 + 1$. When we expand that, we get $h^2 + 6h + 10$. Now we have two points on the parabola. The first point, (x_1, y_1), is equal to (3, 10). The second point, (x_2, y_2), is equal to $(3 + h, h^2 + 6h + 10)$.
3. To calculate the slope of the line that goes through those two points, we have to calculate the change in y divided by the change in x. The symbol used in calculus to express *change in* is delta, Δ. Thus, to calculate the change in y divided by the change in x, we look at $\Delta y/\Delta x$. Algebraically, that's equal to $(y_2 - y_1)/(x_2 - x_1)$.
4. The change in y is $h^2 + 6h + 10 - 10$, and the change in x is $3 + h - 3$. Simplifying, that's $(h^2 + 6h)/h$; when we divide by h, we're left with $h + 6$.
5. That result tells us that the slope of the line that goes through the point (3, 10) and the point very close to (3, 10) is equal to $h + 6$. As we let h get closer to 0, the slope of that line gets closer to 6. When h is 0, $6 + h$ becomes 6; therefore, the slope of the tangent line is 6 when $x = 3$.

D. We could go through the same argument for other points on the parabola. For instance, we could use the same algebra to find the slope of the point $(x, x^2 + 1)$, which is simply $2x$. When $x = 1$, the slope of that tangent line is 2. When $x = -3$, the slope of that tangent line is -6. When $x = 0$, the slope of that tangent line is 0.
   1. In general, for the function $y = x^2 + 1$, the slope at the point x is equal to $2x$, and we represent that with the notation $y' = 2x$. The term for y' is the *slope function* or the *first derivative*.
   2. Note that if we raise or lower the function $y = x^2 + 1$, the tangent line still has the same slope as it did before. If we're looking at the function $x^2 + 17$, or x^2, we still have $y' = 2x$.

E. The official definition of the derivative is as follows: For any function $y = f(x)$, we define y' as $(f(x+h) - f(x))/h$ (that's the change in y divided by the change in x) and we take the limit of that as h goes to 0. Calculating this is called *differentiation*. Other notations for y' include $f'(x)$ and dy/dx.

F. As we just saw, if $y = x^2$, its derivative is $y' = 2x$. By using the same kind of logic we just used, we can come up with some general rules for calculating derivatives.
   1. For example, if $y = x^3$, then $y' = 3x^2$. If $y = x^4$, then $y' = 4x^3$. In general, if $y = x^n$, then $y' = nx^{n-1}$. Even when the exponent is 1,

$y = x$, the derivative would be $1x^0 = 1$. That makes sense because the slope of the line $y = x$ is constantly 1.
   2. We can also multiply by a constant when we're differentiating. For instance, given that $y = x^2$ has the derivative $2x$, then $y = 10x^2$ would have the derivative $10(2x)$, or $20x$.
   3. Here's another simple rule: the derivative of the sum is equal to the sum of the derivatives. For instance, if we know that the derivative of $4x^3 = 12x^2$, the derivative of $8x^2$ is $16x$, the derivative of $-3x$ is -3, and the derivative of 7 (that's a constant function, $y = 7$) is 0, and we want to find the derivative of the sum of all those functions, then we use this rule to get $12x^2 + 16x - 3$.

III. Now that we know how to calculate some derivatives, let's look at what we can do with this knowledge.

A. We begin with the function $y = x^2 - 8x + 10$. Looking at a graph of that function, we might ask: Where is that function minimized? Remember we said that when a function reaches its low point, the slope of the tangent line is 0. Wherever a function reaches its minimum or its maximum—that is, whenever we go from decreasing to increasing or from increasing to decreasing—the slope of the tangent line is 0.
   1. We can find where this function is minimized by finding where the derivative of that function is equal to 0. The derivative of $x^2 - 8x + 10$ is $2x - 8$.
   2. When does that equal 0? Solving $2x - 8 = 0$, we get $x = 4$. That function is minimized when $x = 4$.

B. Let's do another application, this one involving Laurel's Lemonade Stand. For my daughter's lemonade stand, we decided that if she charged x cents per cup, she would sell $(50 - x)$ cups in one day.
   1. If Laurel sells $(50 - x)$ cups, then her revenue is $x(50 - x)$, which is $50x - x^2$. That's the revenue function, which we'll call $R(x)$. The graph of that function is an inverted parabola. Where is that function maximized?
   2. We set the derivative of $50x - x^2 = 0$; thus, $50 - 2x = 0$. That equals 0 when $x = 25$. If Laurel charges 25 cents, she can expect to earn $25(50 - 25)$, or 625 cents, or \$6.25.

C. Laurel's sister, Ariel, wants to create a box where Laurel can keep her supplies. She will make the box, without a lid, from a 12-inch piece of cardboard. To create the box, she cuts four x-by-x squares out of the corners of the cardboard and folds up the edges. What will be the volume of the box?
   1. The volume of a box is length times width times height. If Ariel cuts out an x-by-x square from each of the corners, then the length of each side will be $12 - 2x$; the width will also be $12 - 2x$, and the height when the tabs are folded up is x. Thus, the volume is $(12 - 2x)(12 - 2x)x$; if we expand that, we get $4x^3 - 48x^2 + 144x$, which we call $v(x)$.

2. How can we maximize that volume? We set the derivative of the volume equal to 0. Using the power rule and sum rule, we get $v'(x) = 12x^2 - 96x + 144 = 12(x^2 - 8x + 12)$.
3. Setting this equal to zero, and dividing by 12 gives us $x^2 - 8x + 12 = 0$. We can then factor that polynomial to get $(x - 6)(x - 2) = 0$.
4. Of course, the product of those two numbers can be 0 only if one of the numbers is itself 0. That means either $x - 6 = 0$ or $x - 2 = 0$. Thus, to determine where the volume of the box is maximized, we only need to consider when $x = 6$ or when $x = 2$.
5. We can tell, either by looking at the graph of that function or by actually plugging in the numbers, that when we let $x = 6$, the volume of the box is 0. When $x = 2$, however, we get the biggest volume: $(12 - 2x)(12 - 2x)x = (12 - 4)(12 - 4)2 = 128$ cubic inches.

D. So far, we've solved only problems that involve polynomials, but the power rule is actually even more powerful than it sounds.
1. Again, the rule is that the derivative of x^n is nx^{n-1}, and it works for any exponent n, even negative integers or fractions.
2. For instance, $y = x^{-1}$ is the function $y = 1/x$. The derivative of that, by the power rule, would be $-1(x^{-1-1})$, or $-1(x^{-2})$. In other words, $y' = -1/x^2$.
3. If we were interested in differentiating $y = 1/x^2$, that would be $y = x^{-2}$. If we differentiate that, we get $-2x^{-3}$, or $-2/x^3$. Here's a derivative that we'll see later, $y = \sqrt{x} = x^{1/2}$. If we differentiate that, we get $y' = 1/2(x^{1/2-1}) = 1/2x^{-1/2}$, which equals $\frac{1}{2\sqrt{x}}$.

IV. We might also be interested in differentiating the trigonometric function and the exponential function. Such functions model how sound waves travel or how money grows, and are well worth memorizing.

A. The derivative of the sine function is the cosine function. That is, if $y = \sin x$, then $y' = \cos x$.

B. The derivative of the cosine function is the negative of the sine function. That is, if $y = \cos x$, then $y' = -\sin x$.

C. The most important function in calculus is the function $y = e^x$ because, as mentioned earlier, the derivative of e^x is $y' = e^x$. This function tells us that when we plug in x, not only do we get a value of y, but we also get the slope of the tangent line—how fast that function is changing.

D. The derivative of the natural log of x, $\ln x$, is equal to $1/x$.

V. Let's try to clarify why the derivative of $\sin x$ is $\cos x$.

A. We can look at a graph of the sine function just to see how it increases and decreases. Here, we have the graph of $y = \sin x$. Let's estimate the slope at various points along the graph.

B. For instance, when $x = 0$, the slope of the sine function looks close to 1. At the point $x = \pi/2$, at 90°, we have a slope of 0. Down at π, at 180°, we have a slope of -1. At the bottom of the graph, at $3\pi/2$, we again have a slope of 0. At $x = 2\pi$, we're almost back to where we started, and we again have a slope of 1.

C. The pattern of these slopes, 1, 0, -1, 0, 1…, will repeat forever. If we connect the dots of the slope function, we see that it looks very much like the cosine function.

VI. We know that the derivative of the sum is the sum of the derivatives, but is the derivative of the product the product of the derivatives? Unfortunately, the answer is no. Instead, the *derivative of the product* is "the first times the derivative of the second plus the derivative of the first times the second."

A. The product rule is written as follows: If $y = f(x)g(x)$, then $y' = f(x)g'(x) + f'(x)g(x)$.
1. For example, if we're looking at $y = x^2 \sin x$, we know the derivative of each of x^2 and of $\sin x$, and we can use that to find the derivative of their product.
2. The derivative of the product is the first times the derivative of the second, which would be $x^2\cos x$, plus the derivative of the first times the second, which would be $2x(\sin(x))$. When you add those together, you get the derivative: $x^2\cos x + 2x \sin x$.

B. The quotient rule is shown at right. To remember it, instead of thinking $f(x)/g(x)$, think high over low, or "hi" over "ho." Then, you can remember y' as: ho-di-hi minus hi-di-ho over ho-ho.

> If $y = f(x)/g(x)$, then
> $$y' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

1. For instance, suppose we were constructing an elementary model of planetary motion using a yo-yo moving at constant speed.
2. The tangent of x could tell us the slope of the string when the yo-yo is at time x, and the derivative of the tangent of x could tell us how fast that slope is changing at time x. We want to calculate the derivative of the tangent of x; that's $\sin x/\cos x$. By the quotient rule, $\sin x/\cos x$ has derivative $\frac{\cos x \sin'(x) - \sin x \cos'(x)}{(\cos(x))^2}$, which is $\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$. Thus, $\tan x$ has derivative $1/\cos^2 x$.

VII. As we said at the outset, calculus is the mathematics of how things grow. In general, there are three ways that functions grow.

A. Functions may have a constant growth; those functions are represented by straight lines. Functions may also grow in proportion to their input. For example, a falling body travels faster and faster according to how

long it has been traveling. Finally, functions can grow in proportion to their output. Those functions describe how your bank account grows or how the population grows.

B. All of these functions are described by *differential equations*, which sometimes involve taking derivatives of derivatives, called *second derivatives*. Mathematics, being the language of science, is actually expressed through differential equations. For instance, these equations can describe pendulum motion, vibration, pacemakers, even the beating of your heart. In fact, it's safe to say that, on some levels, your life actually depends on calculus.

Reading:

Colin Adams, Joel Hass, and Abigail Thompson, *How to Ace Calculus: The Streetwise Guide.*

Silvanus P. Thompson and Martin Gardner, *Calculus Made Easy.*

Questions to Consider:

1. A manufacturer wants to create a can that will contain 1 liter of liquid. Use differential calculus to determine the dimensions of the can that will minimize the surface area of the can. (Hint: A cylinder with base radius r and height h has volume $\pi r^2 h$ and surface area $2\pi rh + 2\pi r^2$. CAN you see why?)
2. For the function $y = x^3$, what is the slope of the tangent line that passes through the point (2,8). What is the equation for that line?
3. Find the dimensions of a rectangle with perimeter P that has the largest area.

Lecture Nineteen

The Joy of Approximating with Calculus

Scope: In the previous lecture, we began our study of calculus by calculating the slope of tangent lines for many functions, leading to many interesting applications. In this lecture, we'll see how this simple idea, the slope of the tangent line, has beautiful consequences. We'll also learn a new technique of differentiation called the *chain rule*, which shows how to approximate any function by a polynomial. Because polynomials are the simplest kinds of functions to work with and we know how to differentiate them, this tool is especially useful.

Outline

I. We begin with the chain rule, which refers to chains of functions.

A. We know, for example, that the derivative of $\sin x = \cos x$. We also know that the derivative of $x^3 = 3x^2$. Suppose we want to combine those two functions and calculate the derivative of $\sin(x^3)$.

1. You might guess that the derivative of $\sin(x^3)$ is $\cos(x^3)$ or $\cos(3x^2)$. Both answers are wrong, but they're close. The actual answer is $\cos(x^3)(3x^2)$.
2. In general, if we want to take the sine of $g(x)$ and find the derivative of that function—$\sin(g(x))$—the derivative is equal to $\cos(g(x))\, g'(x)$, or $g'(x)\cos(g(x))$.

B. Let's do another example. Recall that the derivative of the function e^x is still e^x. What about the derivative of e^{x^3}?

1. The chain rule tells us that the answer is e^{x^3} times the derivative of x^3, which is $3x^2$; thus, the derivative we're looking for is $3x^2 e^{x^3}$.
2. In general, $e^{g(x)}$ has derivative $g'(x)\, e^{g(x)}$.
3. We can also improve the first differentiation rule we learned, the power rule. According to this, if $y = x^n$, then the derivative of $x^n = nx^{n-1}$. The derivative of $[g(x)]^n$ would be $n[g(x)]^{n-1}$ times the derivative of $g(x)$. That is, if $y = [g(x)]^n$, then $y' = n[g(x)]^{n-1} g'(x)$.
4. For instance, let's calculate the derivative of $(x^3)^5$. According to the chain rule, that's $5(x^3)^4(3x^2) = 15x^{12}x^2 = 15x^{14}$ as the derivative.
5. We can verify this answer because the problem started off as $(x^3)^5$, which is just an unusual way of writing x^{15}, and we know from the power rule that the derivative of x^{15} is, indeed, $15x^{14}$.
6. In general, the chain rule says that if we have a function of a function, $y = f(g(x))$, then $y' = f'(g(x))\, g'(x)$.

C. Let's now use the chain rule to solve the following cow-culus problem: Claudia the cow is 1 mile north of the X-Axis River, which runs east to west. Her barn is 3 miles east and 1 mile north. She wishes to drink from the X-Axis River, then walk to her barn in such a way as to minimize her total amount of walking. Where on the river should she stop to drink?

1. If she starts at the point (0,1), her barn is at (3,2).
2. Suppose she decides to drink from the point x along the river, that is, at the point $(x,0)$. As she walks from her starting point to x, she creates a right triangle with one leg of length 1, base length x, and hypotenuse length $\sqrt{x^2+1}$.
3. Then, Claudia has to walk from point x to her barn. That's another triangle where the base has length $3-x$, the height is 2, and the hypotenuse is $\sqrt{(3-x)^2+2^2}$. When we expand that, we have $\sqrt{x^2-6x+13}$. The total distance that Claudia walks when she stops at x is $f(x) = (x^2+1)^{1/2} + (x^2-6x+13)^{1/2}$.
4. By the chain rule, $f'(x) = \frac{1}{2}(x^2+1)^{-1/2}(2x) + \frac{1}{2}(x^2-6x+13)^{-1/2}(2x-6)$.
5. We want to find the place where the function $f(x)$ is minimized, and to find such a point, we need to find where the derivative is 0.
6. The solution to this equation is $x = 1$, which we can verify.
7. I gave you the solution of $x = 1$, but how could we have derived it? The fact is that this problem can be solved, if you'll pardon the pun, after just a moment's reflection, without ever using calculus.
   a. Imagine that Claudia, as she walks from her original position to the X-Axis River, instead of walking back to her barn at the coordinates (3, 2), walks to the barn's reflection at the point (3, –2).
   b. Notice that the distance from her drinking point to (3, 2) is the same as the distance from her drinking point to (3, –2).
   c. Since the shortest distance between two points is a straight line, to find the optimal path, we draw a straight line between the original point at (0, 1) to the reflected point at (3, –2). The slope of that line is $-3/3 = -1$. If the line starts at the point (0, 1), then it will hit the x-axis at the point where $x = 1$.

II. Now let's look at a way to approximate the square root of any number in your head. Our tool for this is the all-purpose approximation formula.

A. The all-purpose formula works for almost any differential function. It says: $f(a+h) \approx f(a) + h\,f'(a)$. Generally, the smaller h is, the better the approximation is.

B. The reason this formula works is fairly simple. If we go back to the original definition of the derivative, $f'(a) \approx \dfrac{f(a+h)-f(a)}{h}$. As h goes to 0, that approximation becomes exact. Let's now use this approximation formula to calculate square roots in our heads.

C. We know that the function $f(x) = \sqrt{x}$, has derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$. In particular, if we plug in the value $x = a$, we get $f'(a) = \dfrac{1}{2\sqrt{a}}$.

D. Let's say we want to estimate $\sqrt{106}$. We can break 106 into 100 + 6, and we'll let $a = 100$ and $h = 6$. Our approximation formula tells us that $\sqrt{106} = f(106)$, $\approx f(100) + 6\,f'(100)$. But $f(100)$ is $\sqrt{100} = 10$. To that, we add $6\,f'(100) = \dfrac{6}{2\sqrt{100}} = 6/20$. Hence, our approximation is $10 + 6/20 = 10.3$. As it turns out, $\sqrt{106} = 10.295\ldots$.

E. Let's do another example: $\sqrt{456}$. We know that $\sqrt{400} = 20$, so our first guess is 20 plus an error of 56. We take $20 + 56/2(20)$, which equals $20 + 1.4 = 21.4$.

1. We can get an even better approximation using the process for squaring numbers that we learned in one of our lectures on algebra. Using this process, we know that $21^2 = 441$, which makes our error smaller; h is only 15 instead of 56.
2. In this case, we calculate $\sqrt{456}$ as $21 + 15/2(21) = 21 + 15/42 = 21.357$. The exact answer is 21.354.

III. Let's return to the approximation formula that says $f(a+h) \approx f(a) + h\,f'(a)$.

A. We plug in $a = 0$ and replace h with x to get a much simpler looking equation. This says $f(x) \approx f(0) + f'(0)x$. Once we have the function f, $f(0)$ is just a number, as is $f'(0)$. If $f(x)$ is some number plus some other number times x, that's the equation of a line with a slope of $f'(0)$. That line goes through the point $(0, f(0))$. In other words, we're approximating the function $f(x)$ with a straight line that goes through the same point, $(0, f(0))$, with the same slope.

B. Let's look at the graph of $y = e^x$; near the point (0,1), we have a line (actually, it's the line $y = 1 + x$) that looks just like the function e^x, at least when x is close to 0.

C. If we want an even better approximation, then we look for a parabola, a second-degree polynomial, to go through the same point. Because we have one extra degree of freedom, not only will the parabola go through that point with the same slope (the same first derivative), but the

parabola will also go through that point with the same second derivative.

D. The magic formula for that is $f(x) \approx f(0) + f'(0)x + \dfrac{f''(0)x^2}{2}$. Now we have a parabola that matches the function with the same first derivative and second derivative. We call this the second-degree *Taylor polynomial approximation.*

E. If we want an even better fit, we can get a third-degree approximation by adding a cubic term: $\dfrac{f'''(0)x^3}{3!}$. The reason we use 3! is that now that function will match the original function through the point $(0, f(0))$ with the same first derivative, second derivative, and third derivative.

F. We can also bring this function out to even higher degrees with the same kind of formula.

1. Let's use the function $f(x) = e^x$; we choose this function because $f'(x)$, its first derivative, is e^x. The second derivative is also e^x, as is the third derivative. When we plug those in at 0, $f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$, and $f'''(0) = 1$. That tells us that near the point $x = 0$, e^x is approximately $1 + x + x^2/2! + x^3/3!$.
2. We're approximating the important function e^x by a cubic polynomial, and when we're close to 0, it's a pretty good fit.
3. The n^{th}-degree Taylor polynomial would be $1 + x + x^2/2! + \ldots + x^n/n!$. If we let n go to infinity, we get perfect accuracy for all values of x. This is called the *Taylor series* of x, and it has amazing consequences.
4. For instance, look what happens when we differentiate the Taylor series for e^x (which is $1 + x + x^2/2! + x^3/3! \ldots$), one term at a time: The derivative of 1 with respect to x is 0. The derivative of x is 1. The derivative of $x^2/2!$ is x. The derivative of $x^3/3!$ is $3x^2/3!$, but the 3's cancel and we're left with $x^2/2!$. The derivative of $x^4/4!$ is $4x^3/4!$, which is $x^3/3!$. When we differentiate the terms of the series for e^x, we get e^x again, which makes sense because the derivative of e^x is e^x.

IV. Let's look at some more important Taylor series, which can be derived in the same way that we derived the e^x series.

A. For instance, $\sin x$ has the following Taylor series: $x - x^3/3! + x^5/5! - x^7/7! + x^9/9!\ldots$. This looks just like the odd terms of the e^x series except that the signs alternate. Let's look at the graph of $y = \sin x$ and its approximation with the function $y = x$. The function $x - x^3/3!$ is an even better approximation, and the fifth-order Taylor approximation, $x - x^3/3! + x^5/5!$, is even better.

B. We can figure out the series for $\cos x$ by differentiating the series for $\sin x$. We know that the derivative of $\sin x$ is $\cos x$; differentiating the terms of the Taylor series for $\sin x$, we get the series for $\cos x$, namely $1 - x^2/2! + x^4/4! - x^6/6!\ldots$. Those are the even terms of the e^x series, again with the signs alternating.

V. Now, let's have some more fun with functions.

A. Look at the series for e^{-x}; that's what we get when we take the e^x series and replace all the x's with $-x$. This gives us $e^{-x} = 1 - x + x^2/2! - x^3/3! + x^4/4! - \ldots$. Thus, the e^{-x} series looks like the e^x series except the signs are alternating.

B. If we add the e^x series to the e^{-x} series and divide by 2 (taking the average of those two functions), we get the *hyperbolic cosine function*, or *cosh function*. That is, $\cosh x = (e^x + e^{-x})/2$.

1. Look what happens when we add those series together: The odd terms cancel, and the even terms stay the same. Thus, $\cosh x = 1 + x^2/2! + x^4/4! + x^6/6! + \ldots$.
2. One reason that's called the hyperbolic cosine is that its infinite series looks just like the infinite series of the cosine function except that the cosine function has alternating signs.

C. Similarly, if we subtract those two infinite series, the odd terms survive and the even terms are eliminated. We're left with $\sinh x = (e^x - e^{-x})/2 = x + x^3/3! + x^5/5!\ldots$. It looks just like series for the sine function except that it doesn't have alternating signs. That's called the *hyperbolic sine function*, or *sinh function*.

D. Notice also that $\sinh' x = \cosh x$ and $\cosh' x = \sinh x$.

E. We see hyperbolic functions everywhere in our daily lives. For instance, a hanging cable or piece of rope always fits a cosh curve. In fact, every hanging rope or chain is of the form $y = \dfrac{1}{a}\cosh\left(\dfrac{x}{a}\right)$. Note that to differentiate this function, we would use the "chain rule."

F. Where does the word *hyperbolic* come from in these functions? We know that $(\cos\theta, \sin\theta)$ exists on the unit circle since $\cos^2 + \sin^2 = 1$. Similarly, we can show that $\cosh^2 - \sinh^2 = 1$, which means that $(\cosh\theta, \sinh\theta)$ lies on the unit hyperbola, and that's where the word comes from.

G. Another easy property to verify is that $\cosh x + \sinh x = e^x$; that can be verified by the series or by the original definition.

VI. We've seen a number of parallels between the hyperbolic functions and the trigonometric functions, and if $\cosh + \sinh = e^x$, then there must be some connection among cosine x, sine x, and e^x.

A. The connection is Euler's equation: $e^{ix} = \cos(x) + i\sin(x)$.

B. We could prove that by the series for e^x, replacing all the x's with ix's. As that i is raised to different powers ($i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -1$, $i^4 = 1$) then the sign pattern is: $1, i, -1, -i, 1, i, -1, -i$. As we look at that

pattern and separate the real part from the imaginary part, we get the series for $\cos x$ plus i times the series for $\sin x$. That's the proof of Euler's equation: $e^{ix} = \cos x + i \sin x$.

C. Incidentally, as we observed earlier, when we let $x = \pi$ or 180°, then $e^{i\pi} = -1$, that is, $e^{i\pi} + 1 = 0$. This equation was recently listed as number two on a list in *Physics World* magazine of the 20 greatest equations.

Reading:

Colin Adams, Joel Hass, and Abigail Thompson, *How to Ace Calculus: The Streetwise Guide.*

Silvanus P. Thompson and Martin Gardner, *Calculus Made Easy.*

Questions to Consider:

1. What is the value of $1 - 1/1! + 1/2! - 1/3! + 1/4! - 1/5! + \ldots$?
2. Use the approximation formula to derive a method for mentally determining good approximations to $\sqrt[3]{a+h}$, where a is a number with a known cube root. For example, come up with a good mental estimate of $\sqrt[3]{1024}$.

Lecture Twenty

The Joy of Integral Calculus

Scope: Geometry and trigonometry allow us to compute the area of simple geometrical figures, such as triangles and circles, but how do we measure the area or volume of more irregularly shaped objects? Calculus comes to the rescue. By adding lots of tiny quantities with simple areas (a process known as *integration*), we can find solutions to many practical problems. These calculations are streamlined through the *fundamental theorem of calculus* which fundamentally relates the area under a curve to the curve's *anti-derivative*.

Outline

I. Calculus is typically broken into two parts: differential calculus and integral calculus. Differential calculus, as we've studied in our last two lectures, is the mathematics of how things change and grow. Integral calculus is used, among other things, to calculate areas and volumes.

   A. The big idea in both differential calculus and integral calculus is to calculate quantities associated with curves using quantities associated with straight lines. For example, in differential calculus, we used our understanding of the slope of a straight line to calculate the slopes of parabolas and trigonometric functions.

   B. In integral calculus, where the goal is to calculate areas, we'll begin by looking at areas we understand, such as the area of a rectangle, and use that knowledge to figure out, for example, the area under a curve.

   C. Initially, you wouldn't expect to find much of a connection between the calculation of slopes and the calculation of area, yet those two concepts are intimately connected through the fundamental theorem of calculus. We'll begin this lecture by looking at that theorem.

II. The original problem of integral calculus is to find the area under some kind of a curve, and we can answer questions about that area using the fundamental theorem of calculus.

   A. Suppose we want to carpet a room that is mostly rectangular but has a curved section described by the function $y = f(x)$. According to the fundamental theorem of calculus, to find the area of that region, we first have to find a function, $F(x)$, with $F'(x) = f(x)$. Once we've found that function, we calculate the area with the formula: $F(b) - F(a)$.

   B. Let's look at a specific example, the parabola described by the function $y = x^2$. Suppose we want to find the area under the curve as x goes from 1 to 4. The first step in the fundamental theorem of calculus is to find a function with $F'(x) = x^2$.

1. If we differentiate $x^3/3$, we know from the power rule that we get $3x^2/3$. The 3's cancel and we're left with x^2. Thus, $f(x) = x^3/3$.
2. The next step is to plug in the endpoints to the function we just found. In other words, we calculate $F(4) - F(1) = 4^3/3 - 1^3/3 = 64/3 - 1/3 = 63/3$, which is exactly 21.
3. Therefore, by the fundamental theorem of calculus, the area under the parabola between 1 and 4 is exactly 21. The notation we use for this is shown at right.
4. In this lecture, we'll see how to interpret integrals as infinite sums.

$$\int_1^4 x^2 dx = \left.\frac{x^3}{3}\right]_1^4$$

The symbol $\int$ is an elongated "s," where "s" stands for "sum."

C. Let's do another example. We'll calculate the area under the curve for the function $y = \sin x$ as x goes between 0 and π.
   1. Before we calculate, we do a bit of guessing. We know that the sine function, at its peak, has a height of 1. We could enclose that entire curve inside a rectangle that has a height of 1 and a length of π; thus the area under the curve can't be bigger than π.
   2. To apply the fundamental theorem, we must find a function whose derivative is $\sin x$, and that function is $F(x) = -\cos x$. We then evaluate $F(\pi) - F(0) = -\cos(\pi) + \cos(0) = -(-1) + 1 = 2$; the area under the curve is 2.

D. If we're looking at a curve that goes above and below the x-axis, then we have to interpret the integral slightly differently. For instance, if we're looking at the function $y = \sin(x)$ as x goes from 0 to 2π, then the area below the x-axis is counted negatively.
   1. With that information, what would you expect to find for $\int_0^{2\pi} \sin x dx$? Is there more area above the curve, more area below the curve, or are they equal? Because the function looks symmetrical, we would expect the positive part and the negative part to cancel each other out and give us an answer of 0.
   2. Let's apply the fundamental theorem of calculus to see if we get that answer. The anti-derivative of $\sin x$ is still $-\cos x$. We evaluate this at the endpoints 0 and 2π, but $\cos(2\pi)$ is the same as $\cos 0$, so they cancel each other out exactly. Hence, this integral results in 0, as expected.

III. What is it that makes the fundamental theorem of calculus do its magic?

A. Before we answer that, let's look at a different question: Suppose we have two functions that have the same derivative. Must those two functions be the same? If $f'(x) = g'(x)$, does that mean that $f(x) = g(x)$? The answer is: almost, but not quite.
   1. For example, what functions have the derivative $2x$? We know that x^2 has a derivative of $2x$, as do $x^2 + 1$, $x^2 + 17$, and $x^2 - \pi$.
   2. Anything that's of the form $x^2 + c$ has a derivative of $2x$, and the only functions that have a derivative of $2x$ are of the form $x^2 + c$.
   3. Try to remember this theorem: If two functions have the same derivative, then those two functions differ by a constant. Mathematically, if $f'(x) = g'(x)$, then $f(x) = g(x) + c$.

B. Knowing this theorem, we're ready to answer the question: What makes the fundamental theorem of calculus do its trick? Our goal is to prove that if we have a function $y = f(x)$ and we want to find the area under the curve between the points $x = a$ and $x = b$, we find a function F whose derivative is f, then evaluate $F(b) - F(a)$ to find the area.

C. We begin with the quantity $R(x)$, which is the area of the region under the curve between a and x. Notice that as we vary x, the region under the curve also varies, and its area will vary.
   1. What if we move x on top of a? The area of the region, then, is 0. We're looking at a straight line, which doesn't have any area.
   2. Thus, $R(a) = 0$, as will be useful later.

D. Our goal with the fundamental theorem is to show that the area under the curve from a to b is $F(b) - F(a)$. But, by definition, the area under the curve from a to b, is $R(b)$. Thus, the goal of this theorem is to conclude that $R(b) = F(b) - F(a)$. How are we going to get there?
   1. Remember, $R(x)$ is the area under the curve as we go from a to x. What's $R(x + h)$? By definition, that is the area under the curve as we go from a to $(x + h)$. The difference in those quantities, $R(x + h) - R(x)$, is the area as we go from a to $(x + h)$ minus the area as we go from a to x. Almost everything gets canceled there except for the tiny region between x and $(x + h)$.
   2. Looking at a blowup of that region, we see that if h is really small, the region is almost rectangular, and its area, then, is approximately the area of a rectangle with base h and height $f(x)$; thus, its area is approximately h multiplied by $f(x)$.
   3. Dividing both sides of this equation by h, we get $(R(x + h) - R(x))/h \cong f(x)$. As we let h go to 0, the expression becomes $R'(x) = f(x)$. And since $F'(x) = f(x)$, we have $R'(x) = F'(x)$.
   4. As we said earlier, if two functions have the same derivative, they differ by a constant; therefore, $R(x) = F(x) + c$. That constant must work for every value of x that we plug into it; in particular, it must work when we plug in the value $x = a$.
   5. If we plug in $x = a$, then $R(a) = F(a) + c$. Remember, though, that $R(a) = 0$. Solving for c, we find that $c = -F(a)$. Plugging that value into the formula above, $R(x) = F(x) - F(a)$ for all values of x.

6. Because that works for all values of x, in particular, it must work for $x = b$; therefore, $R(b) = F(b) - F(a)$.

IV. Motivated by the fundamental theorem of calculus, here are some techniques for finding anti-derivatives of functions.

A. We use the following notation for anti-derivatives: $\int f(x)dx$, which represents the set of all functions that have derivative $f(x)$.

B. For example, $\int 2xdx$ is simply asking for all functions that have a derivative of $2x$. We know that all functions with a derivative of $2x$ are of the form $x^2 + c$. Thus, $\int 2xdx = x^2 + c$.

V. Let's look at some other rules for calculating integrals.

A. The power rule for derivatives has a reverse power rule for finding integrals: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$. For example, the reverse power rules says that $\int x^3 dx = x^4/4 + c$. Multiplying through by constants—real numbers—is as easy as it was for derivatives. Since $\int x^3 = x^4/4 + c$, then $\int 7x^3 = 7x^4/4 + c$.

B. Recall that the derivative of the sum was the sum of the derivatives. The same sort of rule works for anti-derivatives. That is, the integral of the sum is the sum of the integrals. For instance, we know $\int 7x^3$ and $\int 2x$; therefore, we can calculate $\int 7x^3 + 2x$ just by adding our previous answers. That would be $7x^4/4 + x^2 + c$.

C. Unfortunately, as we saw with derivatives, the integral of the product is not the product of the integrals. There are some techniques of integration, however, that can help us do these kinds of problems.

1. The equation for a typical bell curve is: $f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$.
2. The bell curve is used to describe numerical quantities, such as exam scores or heights and weights. If we want to find the average value of something that came from a bell-shaped region, then we need to calculate an integral, such as $\int xe^{-x^2} dx$.
3. We'll calculate this integral using the method of integration by guessing. As a guess, we might say it equals e^{-x^2}. By the chain rule, when we differentiate that, we get $(-2x)\, e^{-x^2}$. If it weren't for the -2, we'd have the answer exactly. If we divide through by -2 in our original guess, however, we get: $\int xe^{-x^2} dx = -1/2e^{-x^2} + c$.

D. What if we wanted to find the area between two different points on a bell curve, such as the area between -1 and 2 under the bell curve e^{-x^2}? The fundamental theorem of calculus tells us to find an anti-derivative. Unfortunately, this function, e^{-x^2}, has no simple anti-derivative. We have to resort to the naïve idea of calculating the area by summing up a number of rectangles, at least theoretically.

1. The notation $\int_a^b f(x)dx$ comes from summing a group of rectangles. Imagine breaking up a region from a to b into a bunch of little rectangles. We draw a rectangle that starts at the bottom at the point $(x,0)$ and goes to the top of the curve to the point $(x, f(x))$ with a height of $f(x)$; its base is Δx. The area of that rectangle is $f(x)\Delta x$.
2. If we continue to draw rectangles so that we completely cover that spectrum as x goes from a to b, then we're literally summing values of the form: $\sum_{x=a}^{b} f(x)\Delta x$.
3. As the widths of those rectangles get smaller and smaller, we get $\int_a^b f(x)dx$; thus, when those Δx's go to 0, the Δx becomes a dx.

E. Let's put this into practice by calculating the area of a circle. We can do this simply by adding up the areas of all the little rings inside.

1. The large circle has a radius of R. We extract one ringlet of that circle, which has a radius of r and a circumference of $2\pi r$. We can flatten that ringlet out and look at the area of the edge, whose length is $2\pi r$ and whose thickness is Δr. The total area will be the sum of $2\pi r(\Delta r)$ as the radius goes from 0 to R. As Δr gets smaller, that sum becomes the integral $\int_0^R 2\pi r dr$.
2. We know the anti-derivative of $2\pi r$ is $F(r) = \pi r^2$. Hence the area of a circle is $F(R) - F(0) = \pi R^2 - \pi 0^2 = \pi R^2$, exactly as expected.

VI. The use of the word *integration* in mathematics comes from the fact that we can answer a big problem by breaking it up into smaller, simpler problems, then putting the simple answers together.

A. For example, we can use integration to figure out the volume of a sphere. One way to create a sphere is by taking a flat circle, such as a

lid, and rotating it around the x-axis. Then, we can calculate the volume by chopping the sphere into tiny parts.

B. Chopping off one tiny part, we have a circle with a little bit of thickness, a radius of y, and an area of πy^2. If we call the thickness Δx, then the volume of this small piece is $\pi y^2(\Delta x)$.

C. Because the equation of the original circle was $x^2 + y^2 = R^2$, we can replace y^2 with $R^2 - x^2$. Thus, the sum of $\pi y^2(\Delta x)$ can be written as the sum of $\pi(R^2 - x^2)\Delta x$. We're summing this as x goes from $-R$ to $+R$.

D. In other words, as we let the widths of those slices get smaller, the volume is equal to $\int_{-R}^{R} \pi(R^2 - x^2)dx$. Finding the anti-derivative of that is a fairly simple matter. When we do the algebra, we get $4/3\pi R^3$, which is the volume of a sphere.

VII. Integrals can calculate areas and volume, but also other physical quantities, such as center of mass, energy, and fluid pressure. In fact, along with differential equations, they describe everything from heat to light to sound to electricity. Without a doubt, calculus is an integral part of our daily lives.

Reading:

Colin Adams, Joel Hass, and Abigail Thompson, *How to Ace Calculus: The Streetwise Guide.*

Silvanus P. Thompson and Martin Gardner, *Calculus Made Easy.*

Questions to Consider:

1. Using the chain rule, find the derivative of $\ln x$, $\ln 3x$, and $\ln 7x$. Explain what you see.
2. Verify the calculation expressed by this limerick:

   The integral $z^2\, dz$
   From 1 to the cube root of 3,
   Times the cosine
   Of 3 pi over 9
   Is the log of the cube root of e.

Lecture Twenty-One
The Joy of Pascal's Triangle

Scope: We now turn from calculus to playing with numbers again. As we saw in our lecture on the joy of counting, if we place the binomial coefficients in a triangle, we discover many magical properties that we can explore and derive by counting arguments and the binomial theorem. By summing the rows, columns, and diagonals of the triangle, we discover powers of 2 and hockey sticks. Even the Fibonacci numbers make a surprise guest appearance. Ultimately, we answer the question: In the "Twelve Days of Christmas" song, how many gifts did my true love give to me?

Outline

I. The next three lectures are devoted to topics in probability. We'll use some calculus in these lectures, as well as some discrete mathematics that depends on one of the most beautiful objects in mathematics, Pascal's triangle.

A. Let's begin by looking at the first six rows of Pascal's triangle, labeled 0 through 5. We create numbers in this triangle by adding two consecutive numbers in a given row to produce the number below. These numbers are denoted $T(n, 0)$, $T(n, 1)$, … $T(n, n)$. For instance, in row 4, we have $T(4, 0) = 1$, $T(4, 1) = 4$, $T(4, 2) = 6$, and so on.

0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

B. The rule for creating the rows of Pascal's triangle is: $T(n, 0) = 1$, $T(n, n) = 1$, which says that the row begins and ends with a 1, and for $T(n, k)$, we take $T(n-1, k-1) + T(n-1, k)$. The 10 that appears in row 5 would be known as $T(5, 2)$ and that's equal to $T(4, 1) + T(4, 2)$.

C. We can use this rule to create rows in the triangle. For instance, row 6 would begin with a 1; then the 6 would be obtained by adding 1 + 5. Then, we add 5 + 10 = 15, 10 + 10 = 20, 10 + 5 = 15, 5 + 1 = 6, and we end with a 1 again.

II. Let's take a look at some patterns inside the triangle.

A. For instance, notice that each row is symmetric. It reads the same way left to right as right to left. Formally, we say $T(n, k) = T(n, n-k)$.

B. If we were to add the numbers in the triangle row by row, we see that row 0 adds to 1, row 1 adds to 2, row 2 adds to 4, row 3 adds to 8, and so on. Those are powers of 2; in general, row n sums to the number 2^n, or $T(n, 0) + T(n, 1) + \ldots + T(n, n) = 2^n$.

C. I call the next pattern the *hockey stick identity*. It occurs when we add the diagonals in the triangle. For instance, when we add 1 + 3 + 6 + 10 + 15 + 21 + 28, we get 84, which lies below and to the right of 28. This is the hockey stick identity because of its shape: a long stick that juts out in a new direction to give the next entry of the triangle. This rule works whether we're adding diagonally going to the left or the right.

III. We can understand some of these patterns through combinatorics.

A. Mathematicians typically define $\binom{n}{k}$ as the number of size k subsets of the numbers 1 … n; we defined it as the number of ways to choose k objects from a group of n objects when order is not important. For instance, if I have n students in my class and I need k of them to form a $\binom{n}{k}=\dfrac{n!}{k!(n-k)!}$ committee, then the number of ways to create that committee is $\binom{n}{k}$.

B. We saw the formula for solving this earlier. But for $k<0$ or $k>n$, we don't even think of the formula; we just think of the definition, and we get 0. In other words, how many ways could I create a committee with –5 students? Of course, the answer is 0.

C. How does $\binom{n}{k}$ relate to Pascal's triangle? I claim that $T(n, k) = \binom{n}{k}$.

1. Looking at the first five rows of the triangle, we can see the terms as $\binom{n}{k}$; thus, row 4 (1, 4, 6, 4, 1) is $\binom{4}{0},\binom{4}{1},\binom{4}{2},\binom{4}{3},\binom{4}{4}$.

2. If we calculate $\binom{4}{2}$ by the formula, we get $\dfrac{4!}{2!(2!)}=\dfrac{24}{2(2)}=6$. At least in the first five rows, it looks as if my claim is true. Let's prove this idea.

D. We know that the boundary numbers for $\binom{n}{k}$ satisfy $\binom{n}{0}=1$; $\binom{n}{n}=1$. Thus, the boundary conditions are as expected.

E. The triangle condition was $T(n, k) = T(n-1, k-1) + T(n-1, k)$. Will that growth condition, or *recurrence relation*, remain true as we look at the numbers $\binom{n}{k}$? Can we show that $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$?

F. One way we can show this is true is by using algebra. That is, we add the terms using the factorial definition; we then put those terms over a common denominator of $k!(n-k)!$, add the fractions, and when the dust settles, we get $\dfrac{n!}{k!(n-k)!}$, or $\binom{n}{k}$.

G. We can also use a combinatorial proof. Returning to the original question, from a class of n students, how many ways can I create a committee of size k? On the one hand, we know the answer to that question is $\binom{n}{k}$.

H. On the other hand, we can answer that question through something known as *weirdo analysis*.

1. Imagine that student number n is the weirdo. Among the $\binom{n}{k}$ committees, how many of them do not use the weirdo? We're looking at size k committees from the class of students 1 through $n-1$. By definition, that's $\binom{n-1}{k}$.

2. How many of those committees must use student n? If student n is on the committee, then we must choose $k-1$ more students to be $\sum_{k=0}^{n}\binom{n}{k}=2^n$ on the committee from the remaining $n-1$ students. Again, by definition, we're looking at $\binom{n-1}{k-1}$.

3. There are $\binom{n-1}{k}$ committees without the weirdo and $\binom{n-1}{k-1}$ with the weirdo; their sum is the total number of committees.

4. Hence, the number of size k committees is $\binom{n-1}{k-1}+\binom{n-1}{k}$.

I. Comparing our two answers to the same question, we get $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

J. We've shown that the $\binom{n}{k}$ terms (called *binomial coefficients*) have the same boundary conditions as Pascal's triangle. They will continue to grow in the same way as the entries of Pascal's triangle; therefore, they are the elements of Pascal's triangle.

IV. All the patterns of Pascal's triangle can be expressed in terms of binomial coefficients.

A. For example, let's look at the pattern we saw earlier, that the elements of row n sum to 2^n. In terms of binomial coefficients, this says $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$.

B. We express this idea using *sigma notation*, shown at right. Sigma is the Greek letter Σ and is read: "the sum as k goes from zero to n of…".

Committees of size 0 = $\binom{n}{0}$

Committees of size 1 = $\binom{n}{1}$

Committees of size 2 = $\binom{n}{2}$

…

Total committees = $\sum_{k=0}^{n} \binom{n}{k}$

1. Here's our combinatorial proof, beginning with the question: How many committees can we form from a class of size n? We can break up the question by considering the size of the committee and adding the answers, as shown at right.

2. Now we ask: Why is the number of committees 2^n? We can answer this using the rule of product. To create a committee, we go through the classroom student by student and decide whether or not each student will be on the committee. For each student, we have two choices, on or off, from student 1 up through student n. That's $2 \times 2 \times 2 \times 2 \times 2 \times \ldots \times 2$ n times, or 2^n ways to create a committee.

V. Another useful theorem in mathematics is the *binomial theorem*, which we can find inside Pascal's triangle.

A. Remember this equation from basic algebra: $(x+y)^2 = x^2 + 2xy + y^2$, as appears in row 2 of Pascal's triangle (1, 2, 1). We see that $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, and the coefficients in that expression are row 3 of the triangle (1, 3, 3, 1). The expression $(x+y)^4$ would be $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$, and those coefficients are row 4 (1, 4, 6, 4, 1). In general, for $(x+y)^n$, the coefficients are the numbers in the n^{th} row of Pascal's triangle. Specifically, the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$.

B. We can think of $(x+y)^n$ as $(x+y)(x+y)(x+y)(x+y)\ldots$ n times. There's only one way to get an x^n term, and that's by taking x from the first expression times x from the second expression times x from the third expression, all the way down to x from the last expression.

C. There are n ways to create an $x^{n-1}y$ term simply by deciding which y's we will use, then letting the rest of the terms be x's.

D. For $x^{n-2}y^2$, we choose two terms to be y's and all the rest x's. There are $\binom{n}{2}$ ways to pick two y's here; thus, the coefficient of $x^{n-2}y^2$ is $\binom{n}{2}$.

VI. To summarize, the binomial theorem says: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

This simple formula can be applied to produce many beautiful identities.

A. For example, if we let $x = 1$ and $y = 1$, the binomial theorem tells us that $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$.

B. Here's another identity: $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

1. One way to prove this is to let $y = 1$ in the binomial theorem; thus: $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} = x^k$.

2. Let's differentiate both sides of this equation with respect to x. When we differentiate the left side, we get $n(x+1)^{n-1}$. When we differentiate the right side, each summand has derivative of $\binom{n}{k} kx^{k-1}$. Hence: $n(x+1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} kx^{k-1}$.

3. When we set $x = 1$, all the x's disappear, and we're left with $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

C. We can prove this same theorem combinatorially. For example, from a class of n students, how many ways can we create a committee of any size with a chair?

1. If the committee has size k, there are $\binom{n}{k}$ ways to create the committee. Once we've done that, there are k ways to choose the chair of the committee. Thus, the number of committees of size k with a chair is $\binom{n}{k}k$; the total number of committees over all possible values of k is: $\sum_{k=0}^{n} k\binom{n}{k}$.
2. There's also a more direct way of answering this question. To create a committee of any size from a class of n students, first, we have n ways to pick a chair. Once we've done that, we have to choose a subset of the remaining $n - 1$ students to serve on the committee. How many possible committees can we form from the remaining $n - 1$ students? As we saw earlier, that's 2^{n-1}. The number of committees of this type, then, is $n2^{n-1}$.

VII. Let's look at some other patterns in Pascal's triangle.

A. We summed the rows of the triangle earlier; let's now sum the diagonals of the triangle. We write it as a right triangle to make the pattern easier to see. Summing the first diagonals, we get 1, 1, 2, 3, 5, 8, and so on. These sums are the Fibonacci numbers.

B. In any given row of Pascal's triangle, how many of the numbers are odd? The top row has one odd number, the next row has two, the next row also has two, the next row has four, and so on.

1. The number of odd numbers in each row of Pascal's triangle is always a power of 2. In fact, it's 2 raised to the number of 1's in the binary expansion of n. Let's look at an example of this.
2. Row 81 of Pascal's triangle has the numbers $\binom{81}{0}, \binom{81}{1}, \binom{81}{81}$.

   How many of those binomial coefficients are odd?
3. The number 81, written in terms of powers of 2, is $64 + 16 + 1$, which in binary notation is 1010001. There are three 1's in that binary expansion of 81, so that will be our exponent. The number of odd numbers in row 81 of Pascal's triangle is $2^3 = 8$.
4. The positions of the 8 odd numbers in row 81 of Pascal's triangle are those numbers that can be formed using a subset (possibly empty) of the numbers 1, 16, and 64. They are: 0, 1, 16, 64, $1 + 16 = 17$, $1 + 64 = 65$, $16 + 64 = 80$, and $1 + 16 + 64 = 81$.

VIII. Let's end on a holiday note with "The Twelve Days of Christmas." What is the total number of gifts received by the end of the 12 days?

A. On the k^{th} day, you received $1 + 2 + 3 + \ldots + k$, but we know that's equal to $k(k + 1)/2$, which is also equal to the binomial coefficient $\binom{k+1}{2}$. For example, on the 12^{th} day of Christmas, you receive $1 + 2 + 3 + \ldots + 12$ gifts. That's equal to (12)(13)/2, or 78 gifts; it's also equal to $\binom{13}{2}$.

B. All the numbers of gifts you receive (1, 3, 6, 10) lie on Pascal's triangle. In fact, when we summed those numbers earlier, we got the hockey stick identity.

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{13}{2}$$

In general, if we sum the numbers at right, the hockey stick identity tells us that we get $\binom{14}{3}$ gifts altogether.

C. Calculating, that's (14)(13)(12)/3!, or 364 gifts. By the end of the song, you've received one gift for every day of the year, except Christmas.

Reading:

Arthur T. Benjamin and Jennifer J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proof.*

Benedict Gross and Joe Harris, *The Magic of Numbers*, chapter 6.

Questions to Consider:

1. There are three odd numbers in the first two rows of Pascal's triangle. How many odd numbers are in the first 4 rows, the first 8 rows, and the first 16 rows? Find a pattern. Can you prove it? Also, describe the resulting picture of Pascal's triangle if you remove all the even numbers from it (or simply replace each even number with 0 and replace each number with 1).
2. Choose any number inside Pascal's triangle and note the six numbers that surround it. For example, if you choose the number 15 in row 6, then the six surrounding numbers are 5 and 10 (above it), 6 and 20 (beside it), and 21 and 35 (below it). Now draw two triangles around that number so that each triangle contains three of those numbers. For example, the first triangle would contain 5, 20, and 21, while the second the triangle would contain 10, 6, and 35. Show that the product of both sets of numbers will always be the same. For instance, $5 \times 20 \times 21 = 2{,}100$ and $10 \times 6 \times 35 = 2{,}100$. This is sometimes called the *Star of David theorem*, because the two triangles form a star with the original number in the middle.

Lecture Twenty-Two
The Joy of Probability

Scope: In this lecture, we learn to calculate answers to such questions as: What are the chances that when flipping a fair coin three times, I will get exactly three heads? Although the outcome of any single random event may be hard to predict, by applying the *central limit theorem*, we can forecast what will happen with a large collection of random events. We also look at the concepts of independence and dependence, expected value, and variance. This lecture incorporates many subjects we've looked at previously, including infinite series, calculus, and *e*.

Outline

I. The easiest events to understand are those that have equally likely outcomes.

   A. For instance, the flipping of a coin has two possible outcomes, heads or tails. The probability of either outcome is 1/2. Probability is expressed as a number between 0 (impossible) and 1 (certain).

   B. In rolling a fair six-sided die, there are six possible outcomes, each of which has an equal probability of occurring. The probability of rolling any specific number is 1/6; of rolling an even number is 3/6, or 1/2; and of rolling a number that is 5 or larger is 2/6, or 1/3.

   C. There are eight sequences in which you can flip a coin three times, and each of those sequences is equally likely. Once you've flipped two heads, for example, the chance that the third flip is a head is still 1/2.

      1. There are eight possible equally likely outcomes, but there is only one way to flip three heads, so the probability of that outcome is 1/8. The probability of flipping two heads or one head is 3/8. The probability of flipping all tails is 1/8.
      2. In general, if you flip a coin n times, you have two equally likely possibilities for the first outcome, the second outcome, the third outcome, and the n^{th} outcome; therefore, there are 2^n different ways of flipping the coin n times.
      3. How many of those ways of flipping the coin result in exactly k heads? Among those n coin flips, choose k of them to be heads; the other ones will have to be tails. The number of ways of picking k heads is $\binom{n}{k}$. The probability of flipping k heads is $\binom{n}{k}/2^n$.

II. What is the probability that at least two people in a group of n people will have the same birth month and day? With just 23 people, there is at least a 50% chance that two people in the room will have the same birthday.

   A. To see why that's true, let's answer the negative question: What's the probability that everyone in the room has a different birthday? In other words, what are the equally likely events in this situation?

      1. If we write down lists of birthdays for everyone in the room, how many possible lists could we create? We'd have 365 choices for the first list, 365 choices for the second, and 365 choices for the last. The total number of lists that are possible is 365^n.
      2. How many ways can we create lists in which all the birthdays are different? There would be 365 choices for the first birthday, 364 choices for the second, 363 choices for the third, and so on, down to $366 - n$ choices for the last one. The probability that all those birthdays are different would be $365 \times 364 \times 363 \times \ldots (366 - n)/365^n$. The probability that there's at least one match among those people is $1 - \dfrac{365!}{365^n(365-n)!}$.

   B. If we plug in some numbers, we find that the probability of a birthday match with 10 people is 12%. With 20 people, the probability is greater than 40%, and with just 23 people, the probability is 50.7%. With 100 people, the probability of a birthday match is 99.99996%.

III. The notion of *independence* is important in probability problems.

   A. Two events, A and B, are independent if the occurrence of A does not affect the probability that B will occur. For example, the outcome of a coin flip has no influence on the outcome of a roll of a die.

   B. For independent events, the probability of A and B is the probability of A times the probability of B, or $P(A \text{ and } B) = P(A) \times P(B)$. For example, the probability of flipping heads is 1/2; the probability of rolling a 3 is 1/6. The probability that both events will occur is their product, 1/12.

   C. What's the probability of rolling five 3's in a row? The probability of rolling the first 3 is 1 out of 6, and the probability of rolling each of the other 3's is also 1 out of 6. Because each of those rolls is an independent event, the probability of rolling five 3's in a row is $1/6^5$.

   D. What's the probability of rolling the first five digits of pi in order? Even though this sequence seems more random than the previous one, the probability of rolling this specific sequence is also $1/6^5$.

   E. The probability of rolling the numbers 1, 2, 3, 4, and 5 in that order is $1/6^5$. But if we allow any order, each sequence has a probability of $1/6^5$. We can arrange the numbers 1 through 5 in 5!, or 120, ways. Thus, the probability of rolling 1, 2, 3, 4, 5 in any order would be $5!/6^5$.

   F. If I roll a six-sided die ten times, what's the probability that I will roll a 3 exactly two of those times?

      1. I could roll a 3, then another 3, then 8 numbers that are not 3s. The probability of that sequence is 1/6 for the first 3, 1/6 for the second 3, and 5/6 for each succeeding number that is not a 3. Thus, the

probability of seeing the specific outcome of a 3, followed by a 3, followed by 8 non-3's would be $1/6^2(5/6)^8$.

2. However, there are $\binom{10}{2}$ ways of rolling two 3's., or $\binom{10}{2}$ sequences that have a probability of $1/6^2(5/6)^8$; therefore, the answer to the original question is $\binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8$.

3. This is an example of a *binomial probability problem*, one of the most important kinds of problems that appear in probability. In general, when we perform an experiment, such as flipping a coin, n times, each of those experiments has a success probability of p. The number of successes, such as the number of heads, is x (called a *binomial random variable*, meaning that it has two possibilities). In that situation, $P(x = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

G. Let's look at a *geometric probability question*: Suppose I roll a six-sided die repeatedly until I see a 3. The probability that the first 3 will appear on the 10th roll is $(5/6)^9(1/6)$.

IV. Let's now switch our attention to problems involving *dependence*.

A. For these problems, we need to know the *conditional probability formula*: The probability of A given B is the probability of A and B divided by the probability of B, or $P(A \mid B) = \dfrac{P(A \text{ and } B)}{P(B)}$.

B. Let's say I roll a six-sided die and the outcome of that roll is x. Find the probability that x is equal to 6 given that x is greater than or equal to 4.

1. We know that the probability of getting any particular outcome is 1/6, but the probability that the outcome will be greater than or equal to 4 is 1/3. The formula gives us that same conclusion.

2. According to the formula, the probability that $x = 6$ given that x is greater than or equal to 4 is: $P(x = 6 \mid x \geq 4) = \dfrac{P(x = 6 \text{ and } x \geq 4)}{P(x \geq 4)}$.

3. The numerator has redundant values: $x = 6$ and $x \geq 4$; thus, we can rewrite the numerator as $P(x = 6)$. In the denominator, the probability that $x \geq 4$ is 3/6. Therefore, the probability is $\dfrac{1/6}{3/6} = \dfrac{1}{3}$.

C. What about the probability that x is even given $x \geq 4$? Using the same idea, that's:

$$\frac{P(x \text{ is even and } x \geq 4)}{P(x \geq 4)} = \frac{P(x = 4 \text{ or } 6)}{P(x \geq 4)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

D. If A and B are independent events, the conditional probability formula tells us that the probability of A happening given B happens is the probability of A and B divided by the probability of B. But because A and B are independent, the probability of A and B is the probability of A times the probability of B divided by the probability of B:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A),$$

which agrees with our notion of independence.

V. Another important concept in probability is *expected value*.

A. The expected value of a random variable x, which we denote $E[x]$, is the weighted average value of all the possible values that x can take on. Specifically, $E[x] = \sum kP(x = k)$.

B. Let's say x could take on three values: 0 with a probability of 1/2, 1 with a probability of 1/3, or 2 with a probability of 1/6. $E[x]$ is a weighted average of the numbers 0, 1, and 2, where those weights are the probabilities. In this case, $E[x] = 0(1/2) + 1(1/3) + 2(1/6) = 2/3$.

C. Expected values have some properties that we might…expect. For instance, if a is a constant, $E[ax] = aE[x]$.

D. The expected value of $x + y$ is the expected value of x plus the expected value of y: $E[x + y] = E[x] + E[y]$. That's true even if we add n random variables. In other words, the expected value of the sum is the sum of the expected value: $E[x_1 + x_2 + \ldots + x_n] = E[x_1] + E[x_2] + \ldots + E[x_n]$. That's true for any random variables, independent or not.

E. Now we can apply the expected value of the sum to derive the expected value of a binomial random variable.

1. Suppose I flip a coin n times, each with heads probability p, and x is the number of heads that I get. What is the expected number of heads when I perform this experiment n times?

2. Your intuition might tell you that if p is 1/2, and I flip the coin n times, we expect about half the results to be heads. If the probability of heads is 2/3, then we expect the number of heads to be $2n/3$. Thus, $E[x] = np$. We can derive this using an easy method that looks at each individual coin flip.

3. Here's the easy method: x_i is equal to 1 if the i^{th} flip is heads and 0 if it's tails. In other words, $x_i = 1$ with probability p and $x_i = 0$ with probability $1 - p$. Then, the total number of heads will be $x_1 + x_2 + x_3 + \ldots + x_n$. In other words, we're just counting the 1's.

4. Thus, $E[x_i] = 1(p) + 0(1 - p) = p$.
5. In this way, $E[x]$, which is $E[x_1] + E[x_2] + \ldots + E[x_n]$ (the expected value of the sum is the sum of the expected value) is equal to $p + p + p + p + \ldots + p$, a total of n times, which is np.

VI. *Variance* measures the spread of x.

A. If $E[x] = \mu$ (as in *mean*), then the variance of x (Var(x)) is defined as $E[(x - \mu)^2]$. In other words, the measure of the spread is the expected squared distance from the mean. The standard deviation of x is the square root of that quantity.

B. Here are some handy formulas for variance and standard deviation:

1. Though we defined the variance of x in one way, in practice, it's often easier to calculate it as $E[x^2] - E[x]^2$. For example, in the problem we saw earlier, if the probability that $x = 0$ is 1/2, the probability that $x = 1$ is 1/3, and the probability that $x = 2$ is 1/6, then $E[x^2]$ is a weighted average of all the possible values of x^2, which is $1/2(0^2) + 1/3(1^2) + 1/6(2^2) = 1$. As we saw earlier, $E[x] = 2/3$; thus, $\text{Var}(x) = E[x^2] - E[x]^2 = 1 - (2/3)^2 = 5/9$.
2. Another property of variance that is worth knowing is as follows: If x_1 through x_n are independent random variables, then the variance of the sum is the sum of the variances.

VII. So far, we've been dealing with discrete random variables, questions that have nice integer answers. But many random processes have continuous answers. We can address continuously defined quantities using calculus.

A. We describe the probability of continuous quantities by a *probability density function*, a curve that stays above the x-axis and whose area under the curve is 1. With this function, the probability that x is between a and b is the area under the curve between a and b.

B. Let's use a probability density function of $x^2/9$. This is a legal probability density function since $\int_0^3 \frac{x^2}{9}dx$. The probability that x is between 1 and 2 is $\int_1^2 \frac{x^2}{9}dx = \frac{7}{27}$.

C. Continuous random variables have similar formulas to discrete random variables. For instance, the expected value of x if x is a continuous random variable, instead of being a weighted sum of the possible values of x, is a weighted integral of the possible values of x. Specifically, $E[x]$ is the integral of x times the density function of x with respect to x. Similarly, to find the expected value of x^2, we take the integral of x^2 times the density function of x.

VIII. Perhaps the most important continuous random variable of all is the normal distribution—the original bell-shaped curve.

A. The most famous of these is the bell-shaped curve that has a mean of 0 and a variance of 1, but these curves can have different sizes.

B. The most general bell curve has a mean of μ and a variance of σ^2. This has a rather imposing probability density function: $\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$.

C. Every normal distribution has the following property: The probability that a continuous random variable is within one standard deviation of its mean is about 68%, within two standard deviations, about 95%.

D. The fact that the normal distribution is so common stems from the *central limit theorem*, which says that if we add up many independent random variables, we always get approximately a normal distribution.

1. In the coin flip, where a single coin flip could be heads or tails (probability 1/2), we can show that the expected number of heads in a single coin flip is 1/2. The variance of a single coin flip is 1/4.
2. If we flip a coin 100 times, the expected number of heads is 50. The variance of the number of heads is $100 \times 1/4 = 25$. Thus, the standard deviation is 5.
3. Though we can't predict the outcome of a single coin flip, we've got a good handle on the outcome of 100 coin flips. That is, the outcome has an expected value of 50 and a standard deviation of 5.
4. Since this has an approximate normal distribution, there is about 95% chance that the number of heads will be between 40 and 60, that is, $50 \pm 2(5)$.
5. We'll see how to exploit more of this kind of information in our next lecture on mathematical games.

Reading:

Edward B. Burger and Michael Starbird, *The Heart of Mathematics: An Invitation to Effective Thinking*, chapter 7.

Benedict Gross and Joe Harris, *The Magic of Numbers*, chapters 8–13.

Questions to Consider:

1. If 10 people are each asked to think of a card, what are the chances that at least two of them will think of the same card?
2. In the game of Chuck-a-Luck, you bet $1 on a number between 1 and 6; then, three dice are rolled. If your number appears once, you win $1; if your number appears twice, you win $2; and if your number appears three times, you win $3. (If your number does not appear, you lose $1.) On average, how much should you expect to lose on each bet?

Lecture Twenty-Three
The Joy of Mathematical Games

Scope: Sometimes the probability that something happens is influenced by other given information. For example, the chance that a horse will win a race might change depending on whether it was running on a sunny day or a rainy day. In this lecture, we discuss conditional probability and the *law of total probability*. Applying these concepts and concepts from previous lectures, we analyze the chances of winning roulette and craps and predict the long-term losses from playing these games.

Outline

I. Let's start with horseracing and Harvey, a horse who likes to run in the rain.

A. If it rains tomorrow, Harvey has a 60% chance of winning the race, but if it doesn't rain tomorrow, he has a 20% chance of winning the race. Our notation for this is: $P(\text{win} \mid \text{rain}) = .60$ and $P(\text{win} \mid \text{no rain}) = .20$. The question is: What's the probability that Harvey will win the race? That answer depends on the actual probability that it will rain.

B. The probability that Harvey will win is the weighted average of the probability that he wins when it rains and the probability that he wins when it doesn't rain. If the probability of rain is 50%, the expression is: $P(\text{win}) = (.60)(.50) + (.20)(.50) = .40$. If the probability of rain is 70%, the expression is: $P(\text{win}) = (.60)(.70) + (.20)(.30) = .48$.

C. Suppose that the probability of rain on race day is 99%. Harvey's chances for winning now should be almost 60%. We take a weighted average of 60% and 20%, giving 60% a weight of .99 and 20% a weight of .01. The weighted average of those numbers is .596; thus, Harvey has a 59.6% chance of winning, as our intuition told us.

D. The probability that Harvey will win is governed by the *law of total probability*, which states, in general: If an event B has two possible outcomes, B_1 or B_2, then $P(A) = P(A \mid B_1)\text{P}(B_1) + P(A \mid B_2)P(B_2)$. Similarly, if B has n possible mutually exclusive outcomes, B_1 or B_2 or … B_n, then $P(A) = P(A \mid B_1)\text{P}(B_1) + \ldots + P(A \mid B_n)P(B_n)$.

II. Let's use this formula to analyze the game of craps.

A. To play craps, you roll two dice. Let's call the total of those two dice the number B. If B is 7 or 11, you win immediately. If B is 2 or 3 or 12, you lose immediately. If B is 4, 5, 6, 8, 9, or 10, you keep rolling the dice until you get a sum of B—your original total—or a 7. If a sum of B shows up first, you win, and if a 7 shows up first, you lose.

B. According to the law of total probability, the probability of winning (event A) is $P(A) = P(A \mid B_1)\text{P}(B_1) + \ldots + P(A \mid B_n)P(B_n)$. In craps, the B event is the total of the dice. It's easier to determine the probability of winning at craps overall once we know what the number rolled is, and the law of total probability allows us to break this problem up into more manageable pieces according to the numbers rolled.

III. We'll put all the information we need in a "craps table" (shown at the bottom of the page); then, we can figure out some of these probabilities.

A. How do we find the probability of seeing any particular number?

1. Imagine that one of the dice is green and the other one is red. There are 6 possible outcomes for the green die and 6 for the red die, or $6 \times 6 = 36$ possibilities for the green/red combination.
2. Even though we're only interested in the total of some number between 2 and 12 (and those are not equally likely), we're just as likely to see a green 3 and a red 5 as a green 6 and a red 2. Thus, each of the 36 outcomes has the same probability.
3. Note that there is one way to roll a total of 2. There are two ways to roll a total of 3 (a green 2 and a red 1 or a green 1 and a red 2). All the possible outcomes are listed in the matrix below. To find the number of possible outcomes for each number, we just count the number of times a given number appears out of 36.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

B. Knowing these outcomes, we can now start to fill in our craps table, focusing first on the shaded rows:

B	$P(\text{win} \mid B)$	P(B)	Product
2	0	1/36	0
3	0	2/36	0
4	1/3	3/36	3/108 = .027777…
5	4/10	4/36	16/360 = .044444…
6	5/11	5/36	25/396 = .063131…
7	1	6/36	6/36 = .166666…
8	5/11	5/36	25/396 = .063131…
9	4/10	4/36	16/360 = .044444…
10	1/3	3/36	3/108 = .027777…
11	1	2/36	2/36 = .055555…
12	0	1/36	0

1. For instance, the probability of winning given that $B = 2$ is 0; if you roll a 2, you've lost immediately. The probability of winning if you roll a 3 is also 0, as is the probability of winning if you roll a 12.
2. On the other hand, the probability of winning if your first roll is a 7 is 100%, or 1, as is the probability of winning if $B = 11$.

C. Now we turn to some of the trickier probabilities. For instance, what's the probability of winning given that $B = 4$? There are two ways to answer that question.
1. If the initial roll is 4, you keep rolling the dice until you see either another 4 or a 7. If a 4 shows up before a 7, you win. If a 7 shows up before a 4, you lose. From our matrix, we see that there are three ways out of 36 to roll a 4 and six ways out of 36 to roll a 7.
2. The chance of winning on the next roll after you've rolled a 4 would be 3/36. But what are the chances that you win two rolls after that first roll? You didn't roll a 4 or a 7 on the next roll ($P = 27/36$); then you did roll a 4 or a 7 on the following roll ($P = 3/36$). Multiplying those probabilities, we get $(27/36)(3/36)$.
3. You could win on the next roll—that is, no 7 or 4, no 7 or 4, followed by a 4. That has the probability $(27/36)^2(3/36)$, and so on.

4. This is an infinite series—a geometric series—and we know how to sum those. We factor out 3/36 to get $1 + 27/36 + 27/36^2 + \ldots$, which has the form of a geometric series, $1 + x + x^2 + x^3 + \ldots$, and we know that equals $1/(1 - x)$. When we do the algebra, we get 1/3 as the probability of rolling a 4 before rolling a 7.

D. Another way of answering that question is a bit more intuitive.
1. If we look at our matrix again, we see that there are three ways to roll a 4 and six ways to roll a 7. Thus, there are twice as many ways to roll a 7 as there are to roll a 4; therefore, it would make sense that you would be twice as likely to roll a 7 before you roll a 4.
2. The only numbers that are relevant to winning in the matrix are the three 4's and the six 7's. One of those will be the first number that you roll, and three of those possibilities allow you to win and six cause you to lose. That's why $P(\text{win} \mid B=4) = 3/9 = 1/3$, which agrees with our previous calculation.

E. Let's use this easier method to answer the next question: What's the probability that you win given that $B = 5$? There are four ways to roll a 5 in our matrix, and there are six ways to roll a 7. What's the probability that the next number you roll is a 5 before you roll a 7? Of the ten possibilities, four of them are good and six of them are bad, so the chance will be 4/10.

F. What about the probability that you win given that your initial roll was a 6 or an 8? Now you've got a better chance of winning because there are five ways to win and six ways to lose with each number; your chance of winning is 5/11.

G. Using this information, we now look at our completed craps table. The law of total probability tells us to multiply column 2 and column 3; the product then goes in column 4. To get the total probability of winning, we add up all those products, which gives us 244/495 = .492929…, or a 49.3% chance of winning and a 50.7% chance of losing.

H. If you know the rules of craps, you know that you can bet against the shooter. Every time the shooter loses, you win, except if the shooter's initial roll is a double 6. In that case, the shooter loses, but you don't win or lose; the result is called a *push*. That event adds to your losing probability by $(1/36)(1/2) = 1/72 = .014$, which makes up for the difference in the 49.3% chance of losing and 50.7% chance of winning.

I. Putting these numbers together, the expected value when you bet \$1.00 on craps is as follows: $1(.493) - 1(.507) = -.014$. In other words, if you bet \$1.00, then your expected value is –1.4 cents. That doesn't seem like much, but if you play the game long enough, you'll go broke.
1. The expected value is –1.4 cents. The variance of a single bet is almost \$1.00.
2. If you make 100 bets and on average you lose 1.4 cents for every bet, then after 100 bets, you will be down about \$1.40.

3. The variance of the sum is equal to the sum of the variances, so the variance after 100 bets will be 100, but the standard deviation—the quantity we most care about—is $\sqrt{100}$ = $10.00. Thus, your expected loss is $1.40, but the standard deviation is $10.00. You're probably going to lose, but there's a chance that you'll still be on the positive side after 100 bets.
4. After 10,000 bets, you'll be down $140. Because the standard deviation grows with the square root of the number of bets, it will be about $100. You now have less than a 20% chance of being in the black after 10,000 bets.
5. After 1,000,000 bets, you will be down $14,000 with a standard deviation of 1,000. You will almost certainly be within two standard deviations of your expected loss; thus, you have a 95% chance of being down somewhere between $16,000 and $12,000; there's a 99% chance that you'll be within three standard deviations—somewhere between $11,000 and $17,000 down.

IV. A game that is easier to look at is roulette.

A. In American roulette, we have 18 red numbers, 18 black numbers, and 2 green numbers—the 0 and the double 0. If you bet on red, you win $1.00 with a probability of 18/38; you lose $1.00 with a probability of 20/38. Your expected value here is (18/38) – (20/38) = –2/38 = –.0526. You will be down about 5 1/4 cents for every bet.

B. After 100 bets, you will be down about $5.26 with a standard deviation of $10.00. After 10,000 bets, you will be down $526 with a standard deviation of $100. Thus you will almost certainly be down somewhere between $200 and $800.

V. Let's close with something called the Gambler's Ruin problem.

A. In this problem, with each bet, you win $1.00 with probability p and you lose $1.00 with probability $1 - p$, or q. You begin with d dollars and your goal is to reach n dollars. Let's say d = $60 and n = $100.

B. The Gambler's Ruin theorem has a beautiful formula for figuring out your chance of reaching n dollars without going broke: $\frac{1-(q/p)^d}{1-(q/p)^n}$, as long as $q/p \neq 1$. When $q/p = 1$, which happens when p is 1/2, then the answer is d/n. Let's look at the implications of this formula.

C. If you walk into a casino and play a fair game ($p = 1/2$), what are the chances that you will go from $60 to $100 before reaching $0? The answer is 60%. If the game is fair and you start 60% of the way toward your goal, then you will reach your goal with a probability of 60%.

1. In a game such as craps, however, where your probability of winning is 49.3%, your chance of reaching your goal is about 28%. If your probability of winning is 49% instead of 49.3%, your chance of reaching your goal goes to 19%.

2. If you play a game such as roulette, where your probability of winning is 47.3% on any given play of the game, you have only a 1.3% chance of reaching your goal without going broke.
3. On the other hand, if you know a little bit of gambling theory, you might be able to play blackjack with a 51% probability of winning, which means you can reach your goal with a probability of 93%.
4. The lesson here is that if you're going to gamble, you might as well be smart about it.

Reading:

Martin Gardner, *Martin Gardner's Mathematical Games*.

Edward Packel, *The Mathematics of Games and Gambling*.

Questions to Consider:

1. When dealing cards on a table, what is the probability that an ace will appear before a jack, queen, or king appears?
2. If you are dealt two cards at random from a deck of 52 cards, what is the probability that one of the cards is an ace and the other card is a 10, jack, queen, or king?

Lecture Twenty-Four

The Joy of Mathematical Magic

Scope: In this lecture, we apply some of the skills we've learned to perform various feats of mathematical magic. We begin with the creation of magic squares. A magic square is a box of numbers, designed so that every row, column, and diagonal sums to the same total. We will learn techniques for creating magic squares of various sizes, including a personalized method for creating a magic square based on someone's birthday. We will also learn how to compute cube roots in your head, as well as how to do magic with numbers and cards to impress and amaze.

Outline

I. We begin with a trick that involves phone numbers and seems to be intriguing to many people. You may need a calculator to follow along.

A. Let's call the first three digits of your phone number x and the last four digits y. Here are the steps to follow:

Multiply the first three digits by 80: $80x$.
Add 1: $80x + 1$.
Multiply by 250: $(80x + 1)250$.
Add the last four digits of your phone number:
$(80x + 1)250 + y$.
Add the last four digits again: $(80x + 1)250 + y + y$.
Subtract 250: $(80x + 1)250 + y + y - 250$.
Simplify and divide by 2: $(20{,}000x + 2y)/2$.
Answer: $10{,}000x + y$ = your phone number.

B. When we get to the number $10{,}000x + y$, we're just attaching four 0's to x, then adding the number y, which leaves us with the phone number.

II. Let's now turn to magic squares.

A. We'll create a magic square using my daughter's birthday, December 3, 1998. In the first row, we write: 12, 3, 9, 8. Adding those digits, we get 32. Now, we have to fill out the rest of the square in such a way that every row and every column adds to 32. The result is on the left below.

12	3	9	8
8	9	11	4
9	10	2	11
3	10	10	9

A	B	C	D
C–	D+	A–	B+
D+	C+	B–	A–
B	A– –	D+ +	C

B. All the rows and columns in this square add to 32, as do the diagonals, the square in the middle, the squares in each of the corners, and the corners themselves. In fact, the four corners are the original numbers.

C. To create a birthday magic square of your own, suppose that the original birth date had numbers A, B, C, and D. Begin by writing A, B, C, and D, in every row, column, and diagonal in the arrangement shown on the right above.

D. This kind of magic square, where every row and column has the same four numbers is called a *Latin square*. To make the Latin square a bit more magical, we start with in the lower left-hand corner. We leave the B alone, but we change the C that's in the third row, second column, to $C + 1$ (designated C+). Right now, the first diagonal will not add up correctly, so we fix that by changing A to A–. With D, then, that group adds up correctly. To get all the groups to *balance*, we fill out the rest of the square as shown on the right above.

E. Notice that every row, column, diagonal, and group of four is balanced. We can now go back through this process to fill in the square for the birthday we started with.

III. Here's a mathematical game that was inspired by a TV show: Mathematical Survivor.

A. To keep the game simple, we start with six positive, one-digit numbers. In fact, however, this can be done with any number of numbers, and it will always work. Let's use the first six digits of pi: 3, 1, 4, 1, 5, 9.

B. Choose any two of those six numbers to be removed. If we remove 3 and 5, we're left with 1, 4, 1, 9. To replace the numbers we removed, we multiply the two numbers, add them, then add those two results: $3(5) = 15$, $3 + 5 = 8$, and $15 + 8 = 23$; that becomes the fifth number. Now, we have 1, 4, 1, 9, and 23.

C. We then repeat the process. Let's say we eliminate 1 and 4. We multiply them, add them, then add the results: $1(4) = 4$, $1 + 4 = 5$, $4 + 5 = 9$, leaving the list as 1, 9, 23, and 9.

D. Repeating the process, we remove 9 and 23:

$9(23) = 207$, $9 + 23 = 32$, $207 + 32 = 239$.
The list is now 1, 9, 239. We then remove 1 and 239:
$1(239) = 239$, $1 + 239 = 240$, $239 + 240 = 479$.

Now we're left with just two numbers, and when we go through the process, the result is 4,799. Surprisingly, when we start with 3, 1, 4, 1, 5, 9, no matter what order we eliminate the numbers in, we will always end up with 4,799.

E. We started with the numbers 3, 1, 4, 1, 5, 9. To do the trick, I used numbers that are one greater than the original numbers, in this case, 4, 2, 5, 2, 6, 10. I then multiplied these numbers together, which results in 4,800. From that answer, I subtracted 1 to get 4,799. In general, if we start with the numbers $a_1, a_2, \ldots a_n$, the mathematical survivor will be: $(a_1 + 1)(a_2 + 1) \ldots (a_n + 1) - 1$.

F. How does this work? Suppose you start with the numbers a_1 through a_n; I start with the numbers $a_1 + 1$ through $a_n + 1$. While you're playing your game, I play a much simpler game. That is, whenever you choose two numbers, I also choose the corresponding numbers, but all I do is multiply mine together. At the end of the game, my numbers are simply the product of all the original numbers that I chose.

1. Notice, however, that every time you replace the numbers a and b with $ab + (a + b)$, I replace $(a + 1)$ and $(b + 1)$ with $(a + 1)(b + 1) = ab + a + b + 1$. My new number is one greater than your new number. That means that our lists begin one number apart everywhere, and they remain one number apart everywhere.
2. For example, if you start with 3, 1, 4, 1, 5, 9, I start with 4, 2, 5, 2, 6, 10. When you replace 3 and 5 with 23 by multiplying, adding, and adding, I simply multiply 4 and 6 to get 24. My 24 is one greater than your 23. Term by term, my list of five terms is one greater than your list of five terms, and that will remain true at each step in the problem.
3. Because I know that my list is guaranteed to be the product of my six numbers, 4,800, then you're going to be left with a number that's one less than mine, 4,799.

IV. We've learned to do all kinds of amazing mental calculations in this course; let's see how to do instant cube roots in your head.

A. In order to do this, you first have to memorize a table of the cubes of the numbers 1 through 10. Here's the table:

$1^3 = 1$	$3^3 = 27$	$5^3 = 125$	$7^3 = 343$	$9^3 = 729$
$2^3 = 8$	$4^3 = 64$	$6^3 = 216$	$8^3 = 512$	$10^3 = 1{,}000$

B. Notice that each of the last digits in the cubes is different. Also note that when you cube a number, it ends in the same number (for example, 1^3 ends in 1, 4^3 ends in 4), or it ends with 10 minus that number (for example, 2^3 ends in 8, 8^3 ends in 2).

C. Suppose someone tells you that a two-digit number cubed is 74,088. First, listen for the thousands. In this case, it's 74,000. We know that 4^3 is 64 and 5^3 is 125. That means that 40^3 is 64,000 and 50^3 is 125,000. This cube must lie between 64,000 and 125,000, or between 40^3 and 50^3. That tells us that our answer must be 40-something.

D. Because we know the answer is a perfect cube, all we have to do is look at the last digit of that cube, in this case, 8. Only one number when cubed ends in 8, namely, 2. Thus, the last digit of the original two-digit number had to be 2, and 42 must be the original cube root.

E. Let's do one more example. Suppose I cube a two-digit number and I tell you that the answer is 681,472. Once again, listen for the thousands—that's 681,000. The number 681 is between 8^3 and 9^3, or

512 and 729. That means that the original number must begin with 8. The last digit of the cube is a 2. Only one number when cubed ends in 2, namely, 8. Thus, the original number had to be 88.

V. I'd like to end, finally, with a card trick. I'm not going to give you the secret to this card trick, but I have confidence that ` with all the math you've learned in this course, you will be able to figure it out if you watch it a few times.

A. This trick works with the 10's, jacks, queens, kings, and aces from the deck—20 cards. I begin by shuffling the cards to my heart's content—or your heart's content. When you tell me to stop, I will keep the cards in that order, but I will ask you to choose whether I should turn some of the cards face up or face down or whether I pair up some of the cards and keep them in the same order or flip them. In this way, we "randomize" the cards. Finally, I deal the cards out into four rows of five, but you choose whether I deal each row out from left to right or from right to left.

B. The cards now are in a completely random order. I consolidate the cards by folding the rows together. You choose whether I fold the left edge, the right edge, the top, or the bottom. Recall that when we started this trick, I shuffled the cards to your heart's content. I can tell that your heart was content because if we look at the cards that are now face up, we have here the 10, jack, queen, king, and ace of hearts.

C. I hope in this course you've been able to experience the joy of math, indeed, the magic of math, as much as I have.

Reading:

Arthur Benjamin and Michael Shermer, *Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks.*

Martin Gardner, *Mathematics, Magic, and Mystery.*

Questions to Consider:

1. Suppose I cube a two-digit number and the answer is 456,533. What was the original two-digit number, that is, the cube root?
2. How was the final card trick of this lecture done? Here's a hint: At the beginning of the trick, when did the magician offer the choice of "face up or face down" and when did the magician offer the choice of "keep or flip"? As for why the folding procedure works, you might look at the hint given in the second problem of Lecture Ten.

Bibliography

Reading:

Adams, Colin, Joel Hass, and Abigail Thompson. *How to Ace Calculus: The Streetwise Guide.* New York: W. H. Freeman, 1998. A lighthearted but very clear guide to the concepts and techniques of calculus. Highly readable without too many technical details.

Adrian, Y. E. O. *The Pleasures of Pi, e and Other Interesting Numbers.* Hackensack, NJ: World Scientific Publishing, 2006. A collection of beautiful infinite series and products that often simplify to some function of pi or *e*. Designed in a unique format that lets the reader first marvel over the number patterns before presenting the proofs later in the book.

Barnett, Rich, and Philip Schmidt. *Schaum's Outline of Elementary Algebra*, 3rd ed. New York: McGraw Hill, 2004. The Schaum's outlines emphasize learning through problem solving. This book has 2,000 solved problems and 3,000 practice problems.

Benjamin, Arthur T., and Jennifer J. Quinn. *Proofs That Really Count: The Art of Combinatorial Proof.* Washington, DC: Mathematical Association of America, 2003. Most numerical patterns in mathematics, from Fibonacci numbers to numbers in Pascal's triangle, can be explained through elementary counting arguments.

———, and Michael Shermer. *Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks.* New York: Three Rivers Press, 2006. Learn the secrets of how to mentally manipulate numbers, often faster than you could do with a calculator, and other magical feats of mind.

Blatner, David. *The Joy of Pi.* New York: Walker Publishing, 1997. An entertaining history of the number 3.14159..., filled with numerical facts, trivia, and folklore.

Bonar, Daniel D, and Michael J. Khoury. *Real Infinite Series.* Washington, DC: Mathematical Association of America, 2006. A widely accessible introductory treatment of infinite series of real numbers, bringing the reader from basic definitions to advanced results.

Burger, Edward B. *Extending the Frontiers of Mathematics: Inquiries into Proof and Argumentation.* New York: Key College Publishing, New York, 2007. Artfully crafted sequences of mathematical statements gently guide readers through important and beautiful areas of mathematics.

———, and Michael Starbird. *The Heart of Mathematics: An Invitation to Effective Thinking.* Emeryville, CA: Key College Publishing, 2000. This award-winning book presents deep and fascinating mathematical ideas in a lively, accessible, readable way.

Conway, John H., and Richard K. Guy. *The Book of Numbers.* New York: Copernicus, 1996. Ranging from a fascinating survey of number names, words, and symbols to an explanation of the new phenomenon of surreal numbers, this is a fun and fascinating tour of numerical topics and concepts.

Cuoco, Al. *Mathematical Connections: A Companion for Teachers and Others.* Washington, DC: Mathematical Association of America, 2005. This book delves deeply into the topics that form the foundation for high school mathematics.

Dunham, William. *Journey through Genius: The Great Theorems of Mathematics.* New York: Wiley, 1990. Each of this book's 12 chapters covers a great idea or theorem and includes a brief history of the mathematicians who worked on that idea.

———. *The Mathematical Universe: An Alphabetical Journey through the Great Proofs, Problems, and Personalities.* New York: Wiley, 1994. Similar to this course, this book contains 25 chapters, each devoted to some beautiful aspect of mathematics. Everything from numbers to geometry to logic to calculus appears in this extremely well-written book.

Gardner, Martin. *Aha!: Aha! Insight and Aha! Gotcha.* Washington, DC: Mathematical Association of America, 2006. This is my first recommendation for a young reader. This two-volume collection (which is not part of Gardner's *Mathematical Games* series [below]) contains simply stated problems and puzzles, with diabolically clever solutions. Adults will enjoy it, too.

———. *Martin Gardner's Mathematical Games.* Washington, DC: Mathematical Association of America, 2005. This single CD contains 15 books by Gardner, comprising 25 years of his "Mathematical Games" column from *Scientific American.* The most recent book from this series is *The Last Recreations: Hydras, Eggs, and Other Mathematical Mystifications* (Springer).

———. *Mathematics, Magic, and Mystery.* New York: Dover Publications, 1956. The original classic book on magic tricks based on mathematics. Includes tricks based on simple algebra and geometry to curious properties of numbers.

———. *The Second Scientific American Book of Mathematical Puzzles and Diversions.* Chicago: University of Chicago Press, 1987. A collection of many of Gardner's early writings, including a chapter on some of the magical properties of the number 9.

Gelfand, I. M., and M. Saul. *Trigonometry.* New York: Birkhauser, 2001. A basic, accurate, and easy-to-read introduction to trigonometry.

———, and A. Shen. *Algebra.* New York: Birkhauser, 2002. This algebra book focuses on why things are true and does not simply present a collection of disjointed techniques for the reader to master.

Gross, Benedict, and Joe Harris. *The Magic of Numbers.* Upper Saddle River, NJ: Pearson Prentice Hall, 2004. This book introduces the beauty of numbers, the patterns in their behavior, and some surprising applications of those patterns, while teaching the reader to think like a mathematician.

Kiselev, Andrei Petrovich. *Kiselev's Geometry, Book I: Planimetry.* Alexander [illegible] trans. El Cerrito, CA: Sumizdat, 2006. Assuming only very basic [illegible] of mathematics, Kiselev builds the edifice of geometry from the [illegible] supplying both bricks and mortar in the process. The book is very [illegible] contained.

Koshy, Thomas. *Fibonacci and Lucas Numbers with Applications*. New York: Wiley-Interscience Series of Texts, Monographs, and Tracts, 2001. A comprehensive collection of amazing facts about Fibonacci numbers and related sequences, including applications and historical references.

Livio, Mario. *The Golden Ratio: The Story of Phi, the World's Most Astonishing Number*. New York: Broadway Books, 2002. This book is written for the layperson and delves into the number phi, also known as the golden ratio.

Math Horizons. Magazine published quarterly by the Mathematical Association of America (www.maa.org); aimed at undergraduates with an interest in mathematics, with high quality exposition on a wide variety of mathematical topics, including stories about mathematical people, history, fiction, humor, puzzles, and contests.

Maor, Eli. *e: The Story of a Number*. Princeton, NJ: Princeton University Press, 1994. The history of the number *e*, its many beautiful mathematical properties, and surprising applications.

———. *To Infinity and Beyond: A Cultural History of the Infinite*. Princeton, NJ: Princeton University Press, 1991. This book examines the role of infinity in mathematics and geometry and its cultural impact on the arts and sciences.

———. *Trigonometric Delights*. Princeton, NJ: Princeton University Press, 1998. A very readable treatment of the history and applications of trigonometry.

Meng, Koh Khee, and Tay Eng Guan. *Counting*. River Edge, NJ: World Scientific Publishing, 2002. A user-friendly introduction to counting techniques, accessible at the high school level.

Nahin, Paul J. *An Imaginary Tale: The Story of* $\sqrt{-1}$. Princeton, NJ: Princeton University Press, 1998. The history of the imaginary number *i*, its many beautiful mathematical properties, and surprising applications.

Packel, Edward. *The Mathematics of Games and Gambling*. Washington, DC: Mathematical Association of America, 2006. This book introduces and develops some of the important and beautiful elementary mathematics needed for analyzing various games, such as roulette, craps, blackjack, backgammon, sports betting, and poker.

Paulos, John Allen. *A Mathematician Reads the Newspaper*. New York: Anchor, 1997. What sort of questions does a mathematician think about when reading about current events?

Reingold, Edward M. and Nachum Dershowitz, *Calendrical Calculations: The Millenium Edition*. Cambridge: Cambridge University Press, 2001. A history and mathematical description of every known (and ancient) calendar system.

Ribenboim, Paulo. *The New Book of Prime Number Records*, 3rd ed. New York: Springer-Verlag, 2004. Everything you wanted to know about prime numbers, from their history to open problems.

Selby, Peter H. and Steve Slavin. *Practical Algebra: A Self-Teaching Guide*, 2nd ed. New York: Wiley, 1991. This book is written for people who need a refresher course in algebra, teaching the basic algebraic skills alongside problems with real-world applications.

Thompson, Silvanus P., and Martin Gardner. *Calculus Made Easy*. New York: St. Martin's Press, 1998. Martin Gardner revised this old classic to be accessible to all readers.

Tucker, Alan. *Applied Combinatorics*, 5th ed. New York: Wiley, 2006. An enjoyable and comprehensive introduction to the art of counting, appropriate at the collegiate level.

Velleman, Daniel J. *How to Prove It*. New York: Cambridge University Press, New York, 2006. The book prepares students to make the transition from solving problems to proving theorems. The book assumes no background beyond high school mathematics.

Wapner, Leonard M. *The Pea and the Sun: A Mathematical Paradox*. Wellesley, MA: A. K. Peters, Ltd, 2005. This book provides a very accessible introduction to the notion of infinite sets and their paradoxical consequences, for example, how an object can be rearranged so that its volume increases.

Internet Resources:

Blatner, David. *The Joy of* π. Everything you ever wanted to know about the mysterious number pi. www.joyofpi.com.

Fibonacci Association. Fibonacci fanatics (like the author) may wish to join the Fibonacci Association. www.mscs.dal.ca/Fibonacci.

Knott, Ron. *Fibonacci Numbers and the Golden Section*. The first Web site for fascinating Fibonacci facts and folklore. www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html.

Mudd Math Fun Facts. A collection of beautiful mathematical facts that can be appreciated by math students of all ages. This site is created by my colleague Professor Francis Su of Harvey Mudd College. www.math.hmc.edu/funfacts/.

"Online Mathematics Textbooks." Lists 65 college-level math textbooks that are available online for free! www.math.gatech.edu/~cain/textbooks/onlinebooks.html.

The Prime Pages. This site contains prime number research, records, and resources. http://primes.utm.edu/.

Weisstein, Eric. *Wolfram Mathworld*. This site proclaims itself to be the Web's most extensive mathematics resource, and that is probably not an exaggeration. This site was created, developed, and nurtured by Dr. Eric Weisstein, who began compiling an encyclopedia of mathematics when he was a high school student. http://mathworld.wolfram.com/.

Notes